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Balanced ranking mechanisms ☆

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ABSTRACT

In the private values single object auction model, we construct a *satisfactory* mechanism – a symmetric, dominant strategy incentive compatible, and budget-balanced mechanism. The mechanism converges to efficiency at an exponential rate. It allocates the object to the highest valued agent with more than 99% probability provided there are at least 14 agents. It is also ex-post individually rational. We show that our mechanism is optimal in a restricted class of satisfactory *ranking* mechanisms. Since achieving efficiency through a dominant strategy incentive compatible and budget-balanced mechanism is impossible in this model, our results illustrate the limits of this impossibility.

1. Introduction

How should a group of agents allocate a unit of resource among themselves? For instance, consider the problem of allocating a bequest among a group of potential heirs. Many a times, no will exists. Even when a will exists, disputes arise. Designated estate agents are often employed to resolve bequest related problems. A Wall Street Journal article quotes an expert suggesting the following dispute resolution procedure:

> *In family disputes, Ms. Olsavsky says, one option is to have all the items put up for auction. Family members can bid on what they want. The money goes back to the estate to be divided equally (Coombs, 2013).*

There are a number of other examples: a group of firms sharing time slots on a jointly owned supercomputer (Guo et al., 2011); reallocation of the winning good in a bidding cartel (McAfee and McMillan, 1992); a group of municipalities deciding on the location of a stadium (Cramton et al., 1987). A key feature of these problems is that transfers can be used (either as taxes or subsidies) for resource allocation. However, transfers across agents have to balance – for instance, money raised by auctioning a bequest must be redistributed among the heirs.

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Table 1
Convergence in our mechanism.

No of agents	Probability to the highest valued agent
9	92.3%
10	95%
11	96.2%
12	98.1%
13	98.9%
14	99.4%
15	99.6%
16	99.8%
17	99.9%

We design mechanisms for such problems with the aim of achieving efficiency. Efficiency requires one to allocate the bequest to the highest valued heir. In the standard private values model, where each agent has a value for the unit of resource/object and transfers are allowed with quasilinear utility, the Vickrey auction satisfies three compelling desiderata of a mechanism: (a) dominant strategy incentive compatibility (DSIC), (b) (allocative) efficiency – allocating the object to the highest valuation agent, and (c) ex-post individual rationality. A well-known criticism of the Vickrey auction is that it is not budget-balanced – it collects revenue from the agents, which distorts ex-post efficiency. Green and Laffont (1979) show that this criticism applies to every DSIC and efficient mechanism: no DSIC and efficient mechanism can be budget-balanced. We look for a second-best solution, where we explore the limits of this impossibility result:

How close to efficiency can we get using a DSIC and budget-balanced mechanism?

We require our solution to satisfy symmetry – agents with identical valuation must get the object with equal probability and pay the same amount. Symmetry is a compelling fairness property – for instance, in the bequest allocation problem, an asymmetric mechanism may either be unacceptable to potential heirs or lead to unpleasant lawsuits later on.

We identify a class of DSIC, budget-balanced, and symmetric mechanisms that we call *ranking mechanisms*. A ranking mechanism is one that uses a *ranking* allocation rule, which is specified (for n agents) by n numbers $(\pi_1, \dots, \pi_n)$ between 0 and 1 such that they add up to not more than 1 and $\pi_j \geq \pi_{j+1}$ for each j. For every j, the number π_j is the probability with which an agent with the j-th highest value is allocated the object at any generic profile of values. Our main result is a description of the *r-optimal* mechanism – a DSIC, budget-balanced, and symmetric ranking mechanism that beats every such mechanism in terms of the allocation probability to the highest valued agent.

We show that the probability with which the highest valued agent gets the object in our mechanism converges to 1 at an exponential rate. At every profile of values, our r-optimal mechanism allocates the object to the highest valued agent with more than 99% probability, provided there are at least 14 agents. It is also ex-post individually rational. The welfare generated by the r-optimal mechanism converges to efficiency as the number of agents increase. The nature of convergence is shown in Table 1, where we report on the probability with which the highest valued agent gets the object in our mechanism.

The r-optimal mechanism we identify satisfies ex-post individual rationality. Ex-post individual rationality is a desired property of mechanisms ensuring participation.

Ranking mechanisms contain two familiar DSIC, budget-balanced, and symmetric mechanisms: (i) the mechanism that allocates the object to each agent with equal probability without using any transfers and (ii) the *residual claimant* mechanism in Green and Laffont (1979). The residual claimant mechanism is defined by choosing an agent uniformly at random as a residual claimant and conducting a Vickrey auction among the other agents. The revenue generated from the auction is then given to the residual claimant. We refer to this mechanism as the Green–Laffont (GL) mechanism, and note that at profiles of distinct values, it allocates the object to the highest valued agent with probability $1 - 1/n$ and to the second highest valued agent with probability $1/n$.[1] Our r-optimal mechanism coincides with the GL mechanism if the number of agents is no more than 8 but differs from it significantly for more than 8 agents.

Our analysis is prior-free. We use DSIC as our solution concept. As we discuss later in Section 6, Cramton et al. (1987) show that Bayesian incentive compatible, efficient, and budget-balanced mechanisms satisfying a form of individual rationality exists in our model. While the mechanism they propose require information about beliefs of agents (with common prior assumption), our result shows the level of efficiency that can be achieved using DSIC and budget-balanced mechanisms, thus showing the limits of such a prior-free and robust approach in this problem. Recent literature in mechanism design has been investigating such questions in other models (Moulin, 2009; Carroll, 2015).[2]

[1] This mechanism (and its variants) were discussed in the context of public-good provision problem in Green and Laffont (1979). Later, Gary-Bobo and Jaaidane (2000) formally define this mechanism and study its statistical and strategic properties.

[2] There are two recent papers which also provide foundational results of DSIC mechanisms in the private values single object auction environment. Manelli and Vincent (2010) show that in such models, for every Bayesian incentive compatible mechanism, there is an "equivalent" DSIC mechanism – this

In view of the Green and Laffont impossibility result, comparing efficiency levels of two DSIC, budget-balanced, and symmetric mechanisms is a natural question. The notion we use here compares ranking mechanisms by the probability with which the highest valued agent gets the object. Formally, we show that this notion coincides with a *worst-case* measure of efficiency: the worst-case ratio of welfare generated by a ranking mechanism and efficient level of welfare. In a prior-free environment, such worst-case measures give a very robust method of comparing mechanisms. These measures are widely used to compare algorithms in the computer science literature, and in the algorithmic game theory literature (Cavallo, 2006; Guo and Conitzer, 2009). They are also becoming popular in the mechanism design literature (Moulin, 2009; Carroll, 2015; Massó et al., 2015).

From a technical point, our paper extends the Myersonian approach. Recall that Myerson (1981) provides necessary and sufficient conditions for a mechanism to be DSIC.[3] While the analysis of efficient mechanisms can be done by focusing on Groves (or VCG) mechanisms, the analysis of non-efficient mechanisms is often considered challenging – see for instance (Sprumont, 2013). To analyze DSIC and budget-balanced mechanisms (which will be non-efficient because of the Green–Laffont impossibility result), we extend Myerson's characterization – we give necessary and sufficient conditions for a mechanism to be DSIC, budget-balanced, and symmetric. Our characterization reveals a rich but complex class of such mechanisms.[4] The ranking mechanisms that we consider in this paper are much simpler to describe. Though we do not know if we can improve upon our r-optimal mechanism, by considering more complex mechanisms, the overwhelming (exponential) speed of convergence of our mechanism (as shown in Table 1) implies that we may not be losing out much by restricting attention to ranking mechanisms.

We also provide some evidence that our mechanism is robust to evaluation of satisfactory (ranking) mechanisms by other criteria. First, we show that our mechanism is not Pareto dominated by another satisfactory ranking mechanism. Second, we do an expected welfare analysis of ranking mechanisms using priors. We assume independent and identically distributed priors. For a large class of priors, we show that our r-optimal mechanism generates higher expected welfare than the GL mechanism. Moreover, if the priors are uniform distributions the expected welfare maximizing ranking mechanism coincides with the r-optimal mechanism for *many* values of n (number of agents).

The rest of the paper is organized as follows. We present our model in Section 2. We introduce ranking mechanisms and discuss our main results in Section 3. We do some robustness checks of our approach using other optimization criteria in Section 4. In Section 5, we discuss how our results can be extended to the case where there are multiple units but each agent can be allocated at most one unit. We relate our results to the literature in Section 6 and conclude in Section 7. All the omitted proofs are relegated to Appendix A at the end. To keep the proofs of our results lucid, we present them in a different sequence than the sequence in which corresponding results appear in the main text. Hence, we recommend that the proofs be read after reading the main text. Some proofs are given in a supplementary appendix.

2. The model

We consider the standard single object independent private values model with $N = \{1, \dots, n\}$ as the set of agents. Throughout, we assume that $n \geq 3$ – the $n = 1$ case is trivial and the $n = 2$ case is discussed later. Each agent $i \in N$ has a valuation v_i for the object. If he is given $\alpha_i \in [0, 1]$ of the object, or given the object with probability α_i, and he pays p_i for it, then his net utility is $\alpha_i v_i - p_i$. The set of all valuations for any agent is given by $V \equiv [0, \beta]$, where $\beta \in \mathbb{R}$. A valuation profile will be denoted by $\mathbf{v} \equiv (v_1, \dots, v_n)$.

An **allocation rule** is a map $f : V^n \to [0, 1]^n$, where we denote by $f_i(\mathbf{v})$ the probability of agent i getting allocated the object at valuation profile $\mathbf{v}$. We assume that at all $\mathbf{v} \in V^n$, $\sum_{i \in N} f_i(\mathbf{v}) \leq 1$.

A **payment rule** of agent i is a map $p_i : V^n \to \mathbb{R}$. A collection of payment rules of all the agents will be denoted by $\mathbf{p} \equiv (p_1, \dots, p_n)$. A **mechanism** is a pair $(f, \mathbf{p})$. We require our mechanism to satisfy the following three properties:

- A mechanism $(f, \mathbf{p})$ is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$, we have
$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v_i', v_{-i}) - p_i(v_i', v_{-i}).$$
- A mechanism $(f, \mathbf{p})$ is **budget-balanced (BB)** if for every $\mathbf{v} \in V^n$, we have
$$\sum_{i \in N} p_i(\mathbf{v}) = 0.$$
- A mechanism $(f, \mathbf{p})$ is **symmetric** if for every $\mathbf{v} \in V^n$ and for every $i, j \in N$ with $v_i = v_j$, we have
$$f_i(\mathbf{v}) = f_j(\mathbf{v}), \qquad p_i(\mathbf{v}) = p_j(\mathbf{v}).$$

equivalence is in terms of interim expected utility of agents. This result is extended to other settings in Gershkov et al. (2013). Unlike our work, these papers do not impose budget-balance as a constraint – indeed, these equivalence results do not hold if budget-balance constraint is imposed.

[3] His characterization is for Bayesian incentive compatible mechanisms, but can be straightforwardly adapted to DSIC mechanisms.

[4] In a companion paper (Mishra and Sharma, 2017), we consider a non-ranking DSIC, budget-balanced, symmetric, and ex-post individually rational mechanism that only allocates probability to the top valued agent. This mechanism converges to efficiency at a rate similar to the GL mechanism.

We call a mechanism **satisfactory** if it is DSIC, BB, and symmetric.[5] Symmetry allows us to consider a mild notion of fairness in our mechanism. It also explicitly rules out *dictatorial* mechanisms, where the dictator is given the object for free at all valuation profiles.[6]

An allocation rule f is **satisfactorily implementable** if there exists a $\mathbf{p}$ such that $(f, \mathbf{p})$ is a satisfactory mechanism. We are interested in finding satisfactory mechanisms that are almost efficient in the following sense.

At any valuation profile $\mathbf{v}$, denote by $\mathbf{v}[k]$ the set of agents who have the k-th highest valuation at $\mathbf{v}$. More formally,

$$\mathbf{v}[1] := \{i \in N : v_i \geq v_j \; \forall \, j \in N\}.$$

Having defined $\mathbf{v}[k-1]$, we recursively define $\mathbf{v}[k]$ as

$$\mathbf{v}[k] := \{i \in N \setminus (\cup_{k'=1}^{k-1} \mathbf{v}[k']) : v_i \geq v_j \; \forall \, j \in N \setminus (\cup_{k'=1}^{k-1} \mathbf{v}[k'])\}.$$

Definition 1. An allocation rule f is **efficient** at $\mathbf{v}$ if

$$\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1.$$

An allocation rule f is efficient if it is efficient at all $\mathbf{v} \in V^n$. A mechanism $(f, \mathbf{p})$ is efficient if f is efficient.

The efficiency of a BB mechanism is equivalent to maximizing the total welfare of agents at every profile of valuations. To see this, note that the total welfare of agents at a valuation profile $\mathbf{v}$ from a mechanism $(f, \mathbf{p})$ is

$$\sum_{i \in N} \Big[v_i f_i(\mathbf{v}) - p_i(\mathbf{v}) \Big] = \sum_{i \in N} v_i f_i(\mathbf{v}),$$

where the equality followed from BB. This is clearly maximized by assigning the object to the highest valued agents.

Green and Laffont (1979) show that no DSIC and budget-balanced mechanism can be efficient. Hence, a satisfactory mechanism cannot be efficient. The precise question we are interested in is: *what is the "most" efficient satisfactory mechanism?*

2.1. A prior-free notion to measure efficiency

In view of the Green–Laffont result, we adopt one of the well-known notions to measure efficiency of satisfactory mechanisms. Fix a satisfactory mechanism $\mathcal{M} \equiv (f, \mathbf{p})$. Note that at any valuation profile $\mathbf{v}$ with $v_1 \geq \ldots \geq v_n$, the maximum possible social welfare is v_1, and the social welfare achieved by $\mathcal{M}$ is

$$\sum_{i \in N} v_i f_i(\mathbf{v}).$$

The ratio of these two numbers is a good measure of efficiency at the valuation profile $\mathbf{v}$. More precisely, the number

$$f_1(\mathbf{v}) + \frac{1}{v_1} \Big(\sum_{i \neq 1} v_i f_i(\mathbf{v}) \Big),$$

is a measure of efficiency at the valuation profile $\mathbf{v}$. Here, as in the rest of the paper, we assume $\frac{0}{0} = 1$. Note that such a measure *only* depends on f and not on $\mathbf{p}$ because $(f, \mathbf{p})$ is a budget-balanced mechanism. Now, the *worst-case* of this ratio happens when we minimize this over all $\mathbf{v}$. In particular, for a satisfactory mechanism $\mathcal{M} \equiv (f, \mathbf{p})$, the worst-case efficiency is given by

$$\mu^{\mathcal{M}} = \inf_{\mathbf{v}} \Bigg[f_1(\mathbf{v}) + \frac{1}{v_1} \Big(\sum_{i \neq 1} v_i f_i(\mathbf{v}) \Big) \Bigg].$$

A natural objective is to find a satisfactory mechanism that maximizes this worst-case efficiency. As discussed in the introduction, this is a robust method of comparing efficiency of mechanisms. We apply this notion of comparing efficiency levels of mechanisms in a restricted class of mechanisms that we describe next.

[5] Green and Laffont (1977) use the terminology *satisfactory* mechanism to mean something different. Among other things, their satisfactory mechanisms are DSIC and *efficient* but need not be BB and symmetric. We apologize if this creates a confusion.

[6] A weaker version of symmetry would be to consider *anonymity* of the mechanism with respect to net utilities of the agents – see Sprumont (2013) for a formal definition. We will require our stronger version of symmetry for our main result.

3. Ranking mechanisms

In most of the paper, we focus attention on the following class of simple allocation rules and the corresponding satisfactory mechanisms that can be constructed using such allocation rules. We call them *ranking* allocation rules. Each ranking allocation rule is defined by n numbers $(\pi_1, \ldots, \pi_n)$ where each $\pi_i \in [0, 1]$ and $\sum_{i \in N} \pi_i \leq 1$. Informally, at a generic valuation profile $v_1 > v_2 > \ldots > v_n$, for every k, π_k reflects the probability with which agent k (which has rank k at this profile) gets the object. Notice that this probability does not change across valuation profiles as long as the rank of the agent does not change. This feature makes the ranking allocation rules simple, both from the point of view of practical implementation and analysis. Also, we require every ranking allocation rule to be symmetric, and this means that it allocates the object in a particular way when there are ties in valuations. We clarify this tie-breaking by formally defining the ranking allocation rule first.

Definition 2. An allocation rule f is a ranking allocation rule if it is symmetric (i.e., for every $i, j \in N$ and for every $\mathbf{v}$ with $v_i = v_j$, we have $f_i(\mathbf{v}) = f_j(\mathbf{v})$) and there exists numbers $\pi_i \in [0, 1]$ for all $i \in N$ with $\pi_1 \geq \ldots \geq \pi_n$ and $\sum_{i \in N} \pi_i \leq 1$ such that at every valuation profile $\mathbf{v}$ and every $k \in N$, we have

$$\sum_{i \in \cup_{j=1}^{k} v[j]} f_i(\mathbf{v}) = \sum_{i \in \cup_{j=1}^{k} v[j]} \pi_i.$$

A mechanism $(f, \mathbf{p})$ is a ranking mechanism if f is a ranking allocation rule.

To illustrate the tie-breaking, suppose there are seven agents: $N = \{1, \ldots, 7\}$ and consider a valuation profile $\mathbf{v}$ such that $v_1 = v_2 > v_3 = v_4 = v_5 > v_6 > v_7$. Consider a ranking allocation rule $(\pi_1, \ldots, \pi_7)$. According to the definition, agents 1 and 2 will equally share (due to symmetry) the allocation probabilities $(\pi_1 + \pi_2)$, i.e., each agent gets the good with probability $\frac{\pi_1+\pi_2}{2}$. Then, agents 3, 4, and 5 will equally share the allocation probabilities $(\pi_3 + \pi_4 + \pi_5)$. Finally, agents 6 and 7 get allocation probabilities π_6 and π_7 respectively.

Note that breaking ties in this manner in a ranking allocation rule maintains continuity of total welfare in terms of valuations of agents. For instance, consider the valuation profile discussed in the above example. Consider any arbitrarily close *generic* (with distinct valuations for agents) valuation profile to this valuation profile. The total expected value of agents 1 and 2 in this profile is arbitrarily close to $v_1\pi_1 + v_2\pi_2 = v_1(\pi_1 + \pi_2)$, where the equality follows from the fact that $v_1 = v_2$. Hence, we can maintain continuity of total welfare by giving a total of $(\pi_1 + \pi_2)$ probability to agents 1 and 2. Finally, using symmetry, we equally divide this probability among these two agents. This explains the tie-breaking in the ranking allocation rule.

Even though the ranking allocation rule is a simple class of allocation rules, there is a rich subclass of ranking allocation rules that are satisfactorily implementable. Our focus on this class is purely driven by their tractability and simplicity.

Two well-known ranking allocation rules are satisfactorily implementable. The equal-sharing allocation rule, where each agent gets the object with probability $\frac{1}{n}$ is satisfactorily implementable – no transfers are required for this. The other allocation rule comes from a mechanism proposed by Green and Laffont. Pick an agent i uniformly at random. Run a Vickrey auction among the remaining $N \setminus \{i\}$ agents. Give the revenue from the Vickrey auction to agent i. Since agents are treated symmetrically, the Vickrey auction is DSIC, and by construction, the mechanism is budget-balanced.[7]

A closer look at the Green–Laffont mechanism reveals the following. For valuation profiles with a distinct highest valued agent, it allocates the object to him with probability $(1 - 1/n)$ and shares the remaining probability $1/n$ among the second highest valued agents. For valuation profiles with more than one highest valued agents, it allocates the entire object equally among the highest valued agents. Therefore, given Definition 2, the allocation rule used in the Green–Laffont mechanism is a ranking allocation rule, where

$$\pi_1 = 1 - 1/n, \pi_2 = 1/n, \pi_3 = \ldots = \pi_n = 0.$$

To be precise, this is the allocation rule corresponding to the direct mechanism of the Green–Laffont mechanism.

We now characterize the ranking allocation rules that can be satisfactorily implemented.

Notation. For any two non-negative numbers K and K' with $K \geq K'$, we denote by $C(K, K')$ the number of ways we can choose K' agents from a set of K agents.

Proposition 1. *A ranking allocation rule with probabilities $(\pi_1, \ldots, \pi_n)$ is satisfactorily implementable if and only if*

$$\sum_{k=1}^{n} (-1)^k C(n-1, k-1)\pi_k = 0.$$

[7] Green and Laffont (1979) discuss an even larger class of satisfactory mechanisms where they take out a coalition of "residual claimant" agents with some probability, run the Vickrey auction on the remaining agents, and allocate the revenue of the Vickrey auction to the residual claimants equally. These mechanisms are also ranking mechanisms.

Later, in Theorem 6 in Appendix A, we give necessary and sufficient conditions for a general allocation rule f to be satisfactorily implementable. Those necessary and sufficient conditions are complicated – they involve verifying an infinite system of equations. On the other hand, the necessary and sufficient condition for satisfactorily implementing a ranking allocation rule is a *single equation* given by Proposition 1. This hints that it may be tractable to search over the space of ranking allocation rules.

Now, we adapt our notion of efficiency measure by restricting the class of mechanisms to ranking mechanisms.

Definition 3. A ranking allocation rule $(\pi_1, \dots, \pi_n)$ is **r-optimal** if it is satisfactorily implementable and for any other satisfactorily implementable ranking allocation rule $(\pi'_1, \dots, \pi'_n)$, we have

$$\pi_1 \geq \pi'_1.$$

A ranking mechanism $(f, \mathbf{p})$ is r-optimal if (i) $(f, \mathbf{p})$ is a satisfactory mechanism and (ii) f is r-optimal.

The notion of r-optimality is an indirect way of requiring a mechanism to maximize the value of worst-case efficiency in the class of satisfactory ranking mechanisms. To see this, fix a ranking mechanism $\mathcal{M} \equiv (f, \mathbf{p})$ with allocation probabilities $(\pi_1, \dots, \pi_n)$. Note that

$$\mu^{\mathcal{M}} = \inf_{\mathbf{v}} \Big[\pi_1 + \frac{1}{v_1}\big(\sum_{j \neq 1} \pi_j v_j\big)\Big] = \pi_1 + \inf_{\mathbf{v}} \frac{1}{v_1}\big(\sum_{j \neq 1} \pi_j v_j\big) = \pi_1,$$

where we used the fact that infimum of the above expression occurs when each agent $j \neq 1$ has zero valuation.

Later, in Theorem 6, we shall establish the fact that if f is satisfactorily implementable, then there is a unique $\mathbf{p}$ such that $(f, \mathbf{p})$ is a satisfactory mechanism. As a result, we shall only talk about the r-optimality of an allocation rule – the corresponding r-optimal mechanisms are uniquely defined.

3.1. The main result

In this section, we provide our main result, which identifies an r-optimal allocation rule. To do so, we first propose a subclass of ranking allocation rules. In this subclass, at a generic valuation profile, the top ranked agent is given the object with some probability π_1 and agents ranked 2 to ℓ are given the object with equal probability π_2, where $\pi_1 + (\ell - 1)\pi_2 = 1$. Formally, a two-step allocation rule is defined as follows.

Definition 4. A **two-step ranking** allocation rule is a ranking allocation rule with probabilities

$$(\pi_1, \underbrace{\pi_2, \dots, \pi_2}_{\ell-1}, 0, \dots, 0),$$

where $\pi_1 > \pi_2 > 0$ and $\pi_1 + (\ell - 1)\pi_2 = 1$.

Hence, a two-step allocation rule is uniquely defined by (π_1, ℓ) – ℓ is the number of agents receiving positive probability at a generic valuation profile. The GL allocation rule is a two-step ranking allocation rule with $\pi_1 = 1 - 1/n$ and $\ell = 2$. In Proposition 8 (see Appendix A), we characterize the class of two-step ranking allocation rules that can be satisfactorily implemented – this class requires ℓ to be even and π_1 is determined uniquely given an even value of ℓ.[8]

We are now ready to state the main result of the paper. It shows that there is a two-step ranking allocation rule that is r-optimal, which has exponential convergence to efficiency.

Theorem 1. *There is a two-step ranking allocation rule that is r-optimal. Its allocation probabilities $(\pi^*_1, \dots, \pi^*_n)$ are defined as follows:*

$$\pi^*_i = \begin{cases} 1 - \frac{\ell-1}{C(n-2,\ell-1)+\ell} & \textit{if } i = 1 \\ \frac{1}{C(n-2,\ell-1)+\ell} & \textit{if } i \in \{2, \dots, \ell\} \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\ell \in \arg \min_{2 \leq i \leq (n-1),\ i \text{ even}} \frac{(i-1)}{\Big(C(n-2, i-1) + i\Big)}.$$

[8] The combinatorics in all our results is quite striking. However, it is very difficult to provide any intuition for these results unless one goes over the proofs. We discuss this issue a little more in the Conclusion section.

Remark 1. Though Theorem 1 requires at least three agents, we can easily identify the r-optimal mechanism in the two-agent case. Proposition 1 continues to hold even if $n = 2$. As a result, the only ranking allocation rule that can be satisfactorily implemented are those where both the agents get the object with equal probability. Hence, the unique r-optimal allocation rule is the *equal sharing* allocation rule where both the agents get the object with probability 1/2-transfers are not needed to make this allocation rule satisfactorily implementable.

Remark 2. All our optimality results rely on the fact that the valuation space V of each agent is *rich* – an interval with zero as the lowest valuation. We do not know how to extend these results to a setting where V is an arbitrary interval. However, we stress here that the mechanism we derive in Theorem 1 remains valid for any arbitrary interval V. To see this, consider $V := [L, H]$, where $0 \le L < H$. Note that our results along with the mechanism in Theorem 1 hold true if valuation space is $[0, H]$. Now, consider the restriction of this mechanism to the valuation space $[L, H]$ – such a restriction is well-defined and satisfactory. Thus, our mechanism will have the same efficiency properties when $V := [L, H]$. Of course, this mechanism need not satisfy the optimality property claimed in Theorem 1 – though, we have no counter-examples to show this. In fact, we conjecture that our mechanism will remain optimal even in such type spaces.

3.2. Computations

Besides the optimality of the two-step allocation rule identified in Theorem 1, we want to stress the speed with which it converges to efficiency. Because of combinatorial terms in the denominator of the expression for π_1^*, its convergence to 1 is exponential. We spell out the exact nature of this convergence below.

The exact form of the r-optimal allocation rule will depend on the value of n. Note that the value of ℓ is determined by minimizing the following expression over all even $i \le (n-1)$:

$$\min_{2 \le i \le (n-1),\ i \text{ even}} \frac{(i-1)}{\Big(C(n-2, i-1) + i\Big)}.$$

Routine calculations show that the minimum of this expression occurs when $i = 2$ for $n < 8$. Hence, for $n < 8$, the GL allocation rule is the unique r-optimal allocation rule.

If $n = 8$, the minimum of this expression occurs at $i = 2$ or $i = 4$. If $n \ge 9$, the optimal value of ℓ is varies but lies around $\frac{n}{2}$.

Notation. For any positive real number x, we denote by $\lfloor x \rfloor_e$ the largest even integer smaller than or equal to x.

Proposition 2. *The two-step ranking r-optimal allocation rule identified in Theorem 1 satisfies*

$$\begin{aligned} \ell &= 2 \text{ if } n < 8, \\ \ell &\in \{2, 4\} \text{ if } n = 8, \\ \ell &\in \{\lfloor \frac{n-1}{2} \rfloor_e, \lfloor \frac{n+1}{2} \rfloor_e\} \text{ if } n \ge 9. \end{aligned}$$

Hence, for $n < 8$, the GL allocation rule is the unique r-optimal allocation rule.

The proof of Proposition 2 is in the supplementary appendix.

Proposition 2 shows that for $n = 8$, there are many r-optimal allocation rules. For $\ell = 2$ and $\ell = 4$, we have two two-step ranking allocation rules that are r-optimal. Any convex combination of these two allocation rules will also be r-optimal. Note that ranking rules generated by such convex combinations need not be two-step ranking allocation rules.

Proposition 2 allows us to compute the allocation probabilities of the highest valuation agent using the Pascal triangle in Fig. 1. Each row (starting with the second row) represents a particular value of n, starting with $n = 3$ in the second row. By Proposition 2, $\ell = 2$ if $n < 8$, $\ell = 2$ or 4 if $n = 8$, and $\ell \in \{\lfloor \frac{n+1}{2} \rfloor_e, \lfloor \frac{n-1}{2} \rfloor_e\}$ if $n > 9$. In each row of the Pascal triangle, the entries are $C(n-2, 0), C(n-2, 1), \ldots, C(n-2, n-2)$. Now, the value $C(n-2, \ell-1)$ is highlighted in the orange (lighter shaded) cell of each row.[9] The probability of the highest valuation agent is then easily computed from this and the value of ℓ as: $\frac{C(n-2,\ell-1)+1}{C(n-2,\ell-1)+\ell}$, which is shown to the right of the Pascal triangle.

Note that for $n \ge 14$, the object is allocated to the highest valuation agent with at least 99% probability. The Green–Laffont allocation rule will require at least 100 agents to achieve such probability for the highest valuation agent.

[9] The values in the brown (darker shaded) cells correspond to the entries of the Green–Laffont allocation rule.

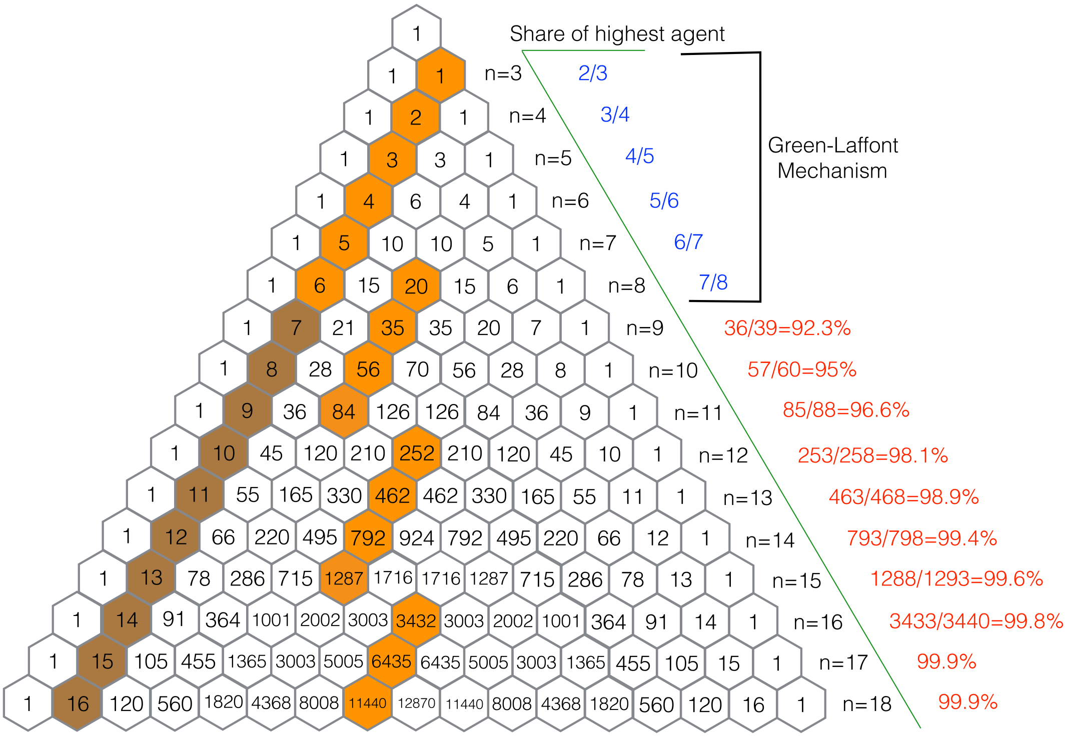

Fig. 1. The r-optimal allocation rule.

3.3. Asymptotic convergence

In this section, we make the exponential convergence of our mechanism to efficiency precise. For every positive integer $n > 2$, Theorem 1 shows that

$$\pi_1^* = 1 - \frac{\ell - 1}{C(n-2, \ell-1) + \ell},$$

where ℓ is chosen according to Proposition 2. We denote, for every $n \in \mathbb{N}$ with $n > 2$,

$$h(n) = \frac{\ell - 1}{(C(n-2, \ell-1) + \ell)}.$$

We show the exponential rate of decay of $h(n)$ to zero. More precisely for any two functions $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of positive integers, we denote $a(n) \sim b(n)$ if

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1.$$

We show below that $h(n) \sim \sigma(n)$, where σ is a function that exponentially converges to zero.

Proposition 3. *For every $n \in \mathbb{N}$,*

$$h(n) \sim \frac{\sqrt{2\pi n}(n-1)}{2^n},$$

where π is the usual irrational number which is the ratio of the perimeter and diameter of any circle.

The proof of Proposition 3 is in the supplementary appendix.

3.4. Participation constraints

We now show that a strong form of participation constraint is satisfied by a class of ranking mechanisms, including the r-optimal mechanism in Theorem 1.

Definition 5. A mechanism $(f, \mathbf{p})$ is **ex-post individually rational** if for every $i \in N$ and for every $\mathbf{v}$, we have

$$v_i f_i(\mathbf{v}) - p_i(\mathbf{v}) \geq 0.$$

The ex-post notion of participation constraint is appropriate in our prior-free model. Notice that, unlike the model in Cramton et al. (1987), our model does not have any property rights defined for the agents.[10] Hence, we assume that the outside option of each agent is zero. In that sense, even though our participation constraints are ex-post, they only ensure non-negative payoff from participation. On the other hand, the participation constraints in Cramton et al. (1987) is interim but because of the property rights structure, they ensure larger interim payoffs to agents.

We prove below that a *class* of mechanisms using two-step ranking allocation rules satisfy ex-post individual rationality. For $n \geq 8$, the two extremes of this class are the Green–Laffont mechanism and our r-optimal mechanism in Theorem 1.

Theorem 2. *Suppose f is a two-step ranking allocation rule defined by (π_1, ℓ), where $2\ell \leq n+1$. If $(f, \mathbf{p})$ is a satisfactory mechanism, then it is ex-post individually rational.*

The r-optimal allocation rule in Theorem 1 satisfies the sufficient condition identified in Theorem 2.

Corollary 1. *Suppose f is the r-optimal allocation rule identified in Theorem 1. If $(f, \mathbf{p})$ is a satisfactory mechanism, then it is ex-post individually rational.*

Proof. By Proposition 2, the r-optimal allocation rule in Theorem 1 satisfies $2\ell \leq n+1$. By Theorem 2, the claim follows. □

We compute the payments in the mechanisms discussed in Theorem 2. While the general payment formula for a satisfactory mechanism is quite complicated (see Theorem 6), the payment formula for the mechanisms in Theorem 2 is easier to express.

Notation. For any pair of positive integers, K, K' with $K \geq K'$,

$$\psi(K', K) := K' \times (K'+1) \times \ldots \times K$$

Proposition 4. *Suppose $(f, \mathbf{p})$ is a satisfactory mechanism, where f is a two-step ranking allocation rule defined by (π_1, ℓ) with $\pi_1 + (\ell - 1)\pi_2 = 1$. For any valuation profile $\mathbf{v}$ with $v_1 > v_2 > \ldots > v_n > 0$, we have*

- *if $i = 1$, then*

$$p_i(\mathbf{v}) = -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=1}^{\ell-1}(-1)^k (k-1)!\,\psi(n-\ell, n-k-1)v_{k+1}\Big].$$

- *if $i \in \{2, \ldots, \ell\}$, then*

$$p_i(\mathbf{v}) = -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{i-1}(-1)^k (k-1)!\,\psi(n-\ell, n-k-1)v_k + \sum_{k=i}^{\ell-1}(-1)^k (k-1)!\,\psi(n-\ell, n-k-1)v_{k+1}\Big].$$

- *if $i > \ell$, then*

$$p_i(\mathbf{v}) = -\frac{\pi_2}{(\ell-1)!}\Big[\sum_{k=2}^{\ell-1}(-1)^k (k-1)!\,\psi(n-\ell, n-k-1)v_k + (-1)^\ell(\ell-1)!v_\ell\Big].$$

The proof of Proposition 4 is in the supplementary appendix.

In any two step ranking allocation rule (π_1, ℓ), at a valuation profile $\mathbf{v}$ with $v_1 > v_2 > \ldots > v_n > 0$, an agent i with $i > \ell$ gets the object with zero probability – call such agents *losing* agents. According to the payment formula computed in Proposition 4, losing agents receive some payments. Theorem 2 shows that losing agents receive non-negative payment if $2\ell \leq n+1$. Hence, participation constraints are satisfied for losing agents in such class of mechanisms. For two step ranking allocation rules, where $2\ell > n+1$, it is possible that losing agents may be asked to pay, violating their participation constraint.

[10] We discuss the results in Cramton et al. (1987) in detail in Section 6.

4. Robustness to other objectives

In this section, we provide some robustness checks for our r-optimal mechanism. In particular, we consider two popular criteria to evaluate mechanisms. In the first criteria, we consider a prior and maximize the expected welfare over the set of all satisfactory ranking mechanisms. We provide a qualitative counterpart of Theorem 1 for this optimization: an expected welfare maximizing ranking mechanism is a 2-step ranking mechanism. The precise parameter of this 2-step ranking mechanism depends on the prior we assume. Next, we use the standard criteria of Pareto domination, where we show that the r-optimal mechanism is not Pareto dominated by any other satisfactory ranking mechanism.

4.1. A prior-based comparison of GL and r-optimal mechanism

In this section, we compare the GL mechanism and the r-optimal mechanism of Theorem 1 using expected welfare criteria. Obviously, we need to define priors for doing such an analysis. We assume independent and identically distributed values. Let F be the probability distribution of valuations of agents drawn from $[0, \beta]$. We assume F is absolutely continuous and admits a positive density function. Using F, we can compute the mean of order statistics – in particular, let G_k be the distribution of the k-th highest valuation. Define the expected k-th highest valuation as

$$\nu_k := \int_0^\beta x dG_k(x).$$

We will call $(\nu_1, \ldots, \nu_n)$ the average order statistics of distribution F.

Pick a ranking allocation rule $(\pi_1, \ldots, \pi_n)$ which is satisfactorily implementable. So, the expected welfare from the ranking allocation rule $(\pi_1, \ldots, \pi_n)$ is

$$\sum_{k=1}^n \pi_k \nu_k.$$

Definition 6. A ranking allocation rule $(\pi_1, \ldots, \pi_n)$ is **expected-welfare optimal (rew-optimal)** if it is satisfactorily implementable and there does not exist another satisfactorily implementable ranking allocation rule $(\pi'_1, \ldots, \pi'_n)$ such that

$$\sum_{k=1}^n \pi'_k \nu_k > \sum_{k=1}^n \pi_k \nu_k.$$

Notice that we are maximizing expected welfare over all *ranking* mechanisms. Below, we give a description of the rew-optimal allocation rule.

Theorem 3. *For every probability distribution F with average order statistics $(\nu_1, \ldots, \nu_n)$, there is a two-step ranking allocation rule that is rew-optimal. Its allocation probabilities $(\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$ are defined as follows:*

$$\tilde{\pi}_i = \begin{cases} 1 - \frac{\ell-1}{C(n-2,\ell-1)+\ell} & \text{if } i = 1 \\ \frac{1}{C(n-2,\ell-1)+\ell} & \text{if } i \in \{2, \ldots, \ell\} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\ell \in \arg \max_{n-1 \geq i \geq 2,\ i \text{ even}} \left[\nu_1 \left(1 - \frac{(i-1)}{\left(C(n-2, i-1) + i\right)} \right) + \frac{1}{\left(C(n-2, i-1) + i\right)} \sum_{j=2}^i \nu_j \right].$$

Theorem 3 is similar in spirit to Theorem 1 – in the class of ranking mechanisms, a simple 2-step ranking mechanism is rew-optimal. But, unlike Theorem 1, it is very difficult to get a precise value of ℓ in Theorem 3 – it will depend on the distribution F (more precisely, on the average order statistics). In particular, it is difficult to say when the rew-optimal mechanism in Theorem 3 will satisfy participation constraints. However, for the uniform distribution, we can be more precise and show that ex-post IR is satisfied.

Proposition 5. *For uniform probability distribution with support $[0, \beta]$, there is a two-step ranking allocation rule that is rew-optimal. Its allocation probabilities $(\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$ are defined as follows:*

$$\tilde{\pi}_i = \begin{cases} 1 - \frac{\ell-1}{C(n-2,\ell-1)+\ell} & \text{if } i = 1 \\ \frac{1}{C(n-2,\ell-1)+\ell} & \text{if } i \in \{2,\ldots,\ell\} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\ell \in \arg \min_{2 \le i \le (n-1):\ i \text{ even}} \frac{i(i-1)}{\big(C(n-2,i-1)+i\big)}.$$

Further, the rew-optimal mechanism also satisfies ex-post individual rationality.

The rew-optimal mechanism described for uniform distribution in Proposition 5 does not coincide with the r-optimal mechanism of Theorem 1. However, their optimal ℓ values can be shown to be close enough. Let ℓ^* be the optimal ℓ in Theorem 1 and $\hat{\ell}$ be the optimal ℓ in Proposition 5. We show in the proof of Proposition 5 that $\hat{\ell} \le \ell^*$. Using steps similar to the proof of Proposition 2, we can also show that $\hat{\ell} \ge \lfloor \frac{(n-3)}{2} \rfloor_e$ – we skip a formal proof of this fact. Hence, even though $\ell^* \neq \hat{\ell}$ for some values of n, they are very close.

Table 2 gives the values of ℓ^* and $\hat{\ell}$ for $n \le 18$. Except for $n = 9, 12, 16$, the values of ℓ^* and $\hat{\ell}$ coincide.

Table 2
r-Optimal and rew-optimal mechanisms for uniform distribution.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
ℓ^*	2	2	2	2	2	2 or 4	4	4	4	6	6	6	6	8	8	8
$\hat{\ell}$	2	2	2	2	2	2	2	2 or 4	4	4	6	6	6	6	8	8

This shows that the rew-optimal mechanism is closer to the r-optimal mechanism than to the GL mechanism for uniform distribution. We now formally show that the r-optimal mechanism in Theorem 1 has better expected welfare than the GL mechanism for a class of distributions satisfying the following condition.

Definition 7. A distribution F with average order statistics $(\nu_1, \ldots, \nu_n)$ satisfies **Assumption A** if for every $j \in \{2, \ldots, n-1\}$, we have

$$\nu_1 - \nu_2 \ge \nu_j - \nu_{j+1}.$$

The uniform distribution on $[0, \beta]$ satisfies Assumption A with equality since $\nu_j - \nu_{j-1} = \frac{\beta}{n+1}$ for all $j \in \{2, \ldots, n\}$.

Proposition 6. *Suppose F is a distribution satisfying Assumption A. Then, for $n \ge 10$, the expected welfare from the r-optimal mechanism in Theorem 1 is as much as the expected welfare from the GL mechanism. Further, for $n > 10$, the expected welfare from the r-optimal mechanism is strictly more than the expected welfare from the GL mechanism.*

Proposition 6 shows that for $n \ge 10$, the r-optimal mechanism is better than the GL mechanism in an expected welfare maximizing sense for a class of distributions (which includes the uniform distribution). This provides another robustness of the r-optimal mechanism.

4.2. Pareto optimal ranking mechanisms

We now discuss an alternate prior-free notion of comparing mechanisms, where we compare mechanisms at *every* valuation profile in term of total social welfare. Informally, a satisfactory mechanism $\mathcal{M}$ dominates another satisfactory mechanism $\mathcal{M}'$ if $\mathcal{M}$ generates as much total welfare as $\mathcal{M}'$ in every profile of valuations and strictly higher in some profile of valuations. A satisfactory mechanism is *Pareto optimal* if it is not dominated by any other satisfactory mechanism.

It is a relatively weak notion to compare mechanisms – for instance, it may be that a Pareto optimal mechanism is dominated by another satisfactory mechanism at a positive measure of valuation profiles. Two satisfactory mechanisms may not even be comparable using this notion.

We adapt the notion of Pareto optimality to the class of ranking mechanisms.

Definition 8. A ranking allocation rule f is **r-Pareto optimal** if (i) f is satisfactorily implementable and (ii) there does not exist another ranking allocation rule f' such that f' is satisfactorily implementable and at every valuation profile $\mathbf{v}$, we have

$$\sum_{i \in N} v_i f'_i(\mathbf{v}) \ge \sum_{i \in N} v_i f_i(\mathbf{v}),$$

with strict inequality holding at some $\mathbf{v}$.

A ranking mechanism $(f, \mathbf{p})$ is r-Pareto optimal if (i) $(f, \mathbf{p})$ is a satisfactory mechanism and (ii) f is r-Pareto optimal.

We first show that the GL allocation rule is an r-Pareto optimal allocation rule.

Theorem 4. *The GL allocation rule is an r-Pareto optimal allocation rule. Moreover, it is the unique r-Pareto optimal allocation rule satisfying* $\pi_3 = \ldots = \pi_n = 0$.

Theorem 4 gives a foundation for the GL mechanism. Among all ranking mechanisms that only allocate the object to top-two agents, the GL mechanism is the unique r-Pareto optimal mechanism. As we show in the next result, if $n \leq 8$, the GL mechanism is the unique r-Pareto optimal mechanism, but there are other r-Pareto optimal mechanisms if the number of agents is greater than 8. In particular, our r-optimal mechanism is always r-Pareto optimal.

Theorem 5. *For* $n \leq 8$*, the GL allocation rule is the unique r-Pareto optimal allocation rule. For* $n > 8$*, the unique r-optimal allocation rule identified in Theorem 1 is also r-Pareto optimal. Further, for any arbitrary r-Pareto optimal allocation rule* $(\pi_1, \ldots, \pi_n)$*, we have*

$$1 - 1/n \leq \pi_1 \leq \pi_1^*,$$

where π_1^* *is as defined in Theorem 1.*

5. A multi-unit extension

Although we focused attention on the single unit allocation case, most of the results can be extended to a multi-unit model as long as agents have unit demand, i.e., each agent can be assigned no more than one unit. We discuss briefly the extension of our results to the multi-unit case – the readers are referred to Long (2016) for formal statements of these results with their proofs.

Consider now a model where a planner has $m < n$ identical units of an object. Each agent can be assigned at most one unit, whose valuation for a unit lies in some interval $[0, \beta]$. Efficiency in this model is equivalent to the following allocation rule: at every valuation profile, allocate the m units to top m valuation agents. Our objective is to find a satisfactory mechanism that has good efficiency property.

The extension of ranking allocation rules to this model is straightforward. It can be defined by n numbers $(\pi_1, \cdots, \pi_n)$ with each $\pi_i \in [0, 1]$ and $\sum_{i \in N} \pi_i \leq m$. The necessary and sufficient condition for satisfactorily implementing a ranking allocation rule is again a single equation, similar to the one given in Proposition 1. Furthermore, maximizing worst-case efficiency in the class of satisfactory ranking mechanisms is equivalent to maximizing the total probability that is assigned to the top m valuation agents, i.e., to maximize $\sum_{1 \leq i \leq m} \pi_i$.

This means that with a suitable modification of the linear program mentioned in the proof of Theorem 1, we can find the r-optimal ranking allocation rule for the multi-unit case. However, unlike the single unit case, the structure of the optimal solution becomes quite involved. For $m \leq \lfloor \frac{n}{2} \rfloor$, the following ranking allocation rule π^* is r-optimal:

$$(\underbrace{1, \cdots, 1}_{m-1}, \pi_m^*, \underbrace{\pi_{m+1}^*, \cdots, \pi_{m+1}^*}_{\ell^* - m}, 0, \cdots, 0),$$

where $\pi_m^* > \pi_{m+1}^* > 0$, $\pi_m^* + (\ell^* - m)\pi_{m+1}^* = 1$, and ℓ^* ($\ell^* \leq \frac{n+1}{2}$) is a minimizer of a function that has similar combinatorics terms as the one in Theorem 1. Although this is not a two-step ranking allocation rule, it allocates a unit each to top $(m-1)$ agents (with probability 1), allocates m-th ranked agent with some probability, and the remainder probability is shared between the next $(\ell^* - m)$ agents.

Table 3
An illustration of r-optimal allocation rule for $n = 9$ and $m = 8$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
$m = 1$	$\frac{12}{13}$	$\frac{1}{39}$	$\frac{1}{39}$	$\frac{1}{39}$	0	0	0	0	0
$m = 8$	1	1	1	1	1	$\frac{38}{39}$	$\frac{38}{39}$	$\frac{38}{39}$	$\frac{1}{13}$

However, for $m > \lfloor \frac{n}{2} \rfloor$, an r-optimal allocation rule has a different structure. It can be deduced directly from the $m \leq \lfloor \frac{n}{2} \rfloor$ case given above. First, compute the r-optimal allocation rule for $(n - m) \leq \lfloor \frac{n}{2} \rfloor$ units case. For instance, consider $n = 9$, $m = 8$, and Theorem 1 can be used to find the allocation probabilities for $n - m = 1$ unit case – see first row in Table 3. If we denote this allocation probability of k-th ranked agent (when there are $(n - m)$ units) as $\pi_{k,n-m}$, then the r-optimal allocation rule for m units assigns a probability of $1 - \pi_{n-k,n-m}$ to the k-th ranked agent. The second row in Table 3 shows the r-optimal allocation rule – note that the second row is just 1 minus the first row written in reverse order. The intuition for this "duality" is difficult to explain and is hidden in the combinatorics of the computations – interested readers are referred to Long (2016).

For $m \leq \lfloor \frac{n}{2} \rfloor$, the r-optimal mechanisms described above are ex-post individually rational. However, it is not always the case for larger m – see Long (2016) for a detailed analysis. Finally, the asymptotic convergence result extends to the multi-unit case, i.e., for fixed m, the r-optimal mechanisms converges to efficiency at an exponential rate. Summarizing, though the qualitative nature of the result extends to the multi-unit unit-demand model, there are many twists and turns.

6. Relation to the literature

The impossibility of achieving efficiency, dominant strategy incentive compatibility, and budget-balance was first shown by Green and Laffont (1979), which also contains a lot of discussions on achieving *second-best* using non-efficient but DSIC and budget-balanced mechanisms. This includes the Green–Laffont mechanism that we discuss. Though, they focussed attention on public good problems and gave sketches of the Green–Laffont mechanism we discuss, they clearly anticipated the mechanism as well as many environments beyond the public good problem where the impossibility result would hold. Gary-Bobo and Jaaidane (2000) contains an extensive discussion on this – they also formally define the Green–Laffont mechanism and study its statistical and strategic properties in the public good problem.

This impossibility result started a long literature on how to overcome it. We classify them in several categories and discuss some relevant ones. Most of the literature we discuss is concerned with private good allocation among several buyers. There is a parallel literature on bilateral trading and public good provision that we do not discuss.

Domain identification. Classic revenue equivalence results imply that every efficient and DSIC mechanism must be a Groves mechanism (Green and Laffont, 1977; Holmström, 1979). The Green–Laffont impossibility result essentially implies that no Groves mechanism can balance budget in many settings – though their focus is mainly of public good problems. In the public good context, Laffont and Maskin (1980) consider differentiable mechanisms and show that existence of a DSIC, BB, and efficient mechanism is equivalent to solving a system of differential equations. In the same model, Walker (1980) identifies domains (of utility functions of agents) where impossibilities exist – he restricts attention to continuous mechanisms. As a corollary of their results, they identify forms of utility functions of agents where possibility or impossibility result exists. Hurwicz and Walker (1990) extend the Green–Laffont impossibility to pure exchange economies. These papers are mainly focused on identifying domains where the negative result of Green and Laffont persists.

But there are settings where DSIC, BB, and efficient mechanisms exist. Suijs (1996) is a good example of a domain where Groves mechanisms that balance the budget exists – he discusses a *sequencing problem*. In the context of multi-object assignment, a recent contribution is Mitra and Sen (2010). This paper identifies domains of multi-object auctions where the Green–Laffont impossibility can be overcome.

Bayesian incentive compatibility. One way to get around the Green–Laffont impossibility is to consider the weaker solution concept of Bayesian incentive compatibility. Arrow (1979); d'Aspremont and Gérard-Varet (1979) construct Bayesian incentive compatible, efficient, and budget-balanced mechanism, now known as the dAGV mechanism, that work in a variety of settings. The dAGV mechanisms fail to be interim individually rational in many settings. In an unpublished paper, Fudenberg et al. (1995) extend this result in the following sense – for every Bayes–Nash implementable allocation rule, there exists a Bayesian incentive compatible and budget-balanced mechanism using this allocation rule. Like in the dAGV mechanism, such budget-balanced mechanisms need not satisfy interim individual rationality. Rahman (2011) gives a characterization of Bayesian (and ex-post) incentive compatible and budget-balanced mechanisms in a very general framework.

In a seminal paper, Cramton et al. (1987) show that efficient, Bayesian incentive compatible, budget-balanced mechanisms satisfying interim individual rationality is possible in a single object allocation problem.[11] The possibility result in our problem using Bayesian incentive compatibility is in sharp contrast to the impossibility results known in bilateral trading problems like in Myerson and Satterthwaite (1983).

Unlike Cramton et al. (1987), we focus on DSIC mechanisms, and our mechanism is not efficient. Naturally, the mechanism in Cramton et al. (1987) require a lot of prior information. Our mechanism is prior-free and satisfies ex-post individually rationality. Thus, we illustrate a prior-free way of approximately achieving the possibility result in Cramton et al. (1987).

Redistribution mechanisms. The prior-free approach of mechanism design using DSIC mechanisms have been popular in algorithmic game theory literature in computer science. Restricting attention to efficient mechanisms, which means restricting attention to Groves mechanisms, several papers relax budget-balance and show how best to redistribute the surplus revenue. The measure of efficiency of redistribution is *worst-case* in these papers. One of the earliest papers to do this is Cavallo (2006), who studied this problem in our setting (single object allocation). He showed that remarkable Groves mechanisms exist that can redistribute large fraction of Vickrey auction payments using Groves mechanisms. Moulin (2009) and Guo and Conitzer (2009) derive optimal redistribution mechanisms in the multi-unit allocation setting where agents demand exactly one unit – their mechanisms are identical and discovered independently.[12] As the number of agents increase, like our mechanism, their Groves mechanisms can redistribute large fraction of Vickrey auction revenue among agents. The main difference from these papers and ours is budget-balance. Since these papers do not impose budget-balance, the actual budget imbalance in these mechanisms can be high in various valuation profiles. On the other hand, like in Cramton et al. (1987), budget-balance is a constraint in our problem. Hence, unlike these papers, we work with mechanisms outside the

[11] They consider a problem where agents have property rights over the object, and stronger form of interim individual rationality is satisfied by their mechanism. However, their results can still be applied to our problem if we assume equal property rights to all the agents.

[12] Several papers related to this theme have also appeared – see for instance Apt et al. (2008) and Moulin (2010).

Groves class. Our results show that we can achieve excellent levels of efficiency (99% with at least 14 agents) using DSIC and budget-balanced mechanisms.

Beyond Groves mechanisms. While most of the literature seems to have either weakened DSIC to Bayesian incentive compatibility or relaxed the budget-balanced criteria while working with efficient and DSIC mechanisms (Groves mechanisms), there is very little literature on exploring the limits of DSIC and budget-balanced mechanisms. We do this for the case of single object allocation problem. One of the problems with exploring non-Groves mechanisms is that we search over the space of allocation rules and payment rules – Groves mechanisms pin down the allocation rule to be the efficient allocation rule. A non-efficient allocation rule can achieve better social welfare redistribution is well known – see for instance examples in Laffont and Maskin (1980) and a more computational analysis in de Clippel et al. (2009). Sprumont (2013) considers Pareto-undominated mechanisms by considering DSIC and non-efficient mechanisms, though his mechanisms are not budget-balanced. Faltings (2005) and Guo et al. (2011) consider variants of Green–Laffont mechanisms discussed in Green and Laffont (1979) and show some worst-case results, but they do not consider the general class of DSIC and budget-balanced mechanisms that we analyze. Hashimoto (2015) discusses a non-ranking satisfactory mechanism and provides several axiomatization his mechanism.

Another possibility is to consider priors and design the expected welfare maximizing DSIC and budget-balanced mechanism for allocating an object. This is similar to the expected revenue maximizing question in Myerson (1981), but significantly more complicated. Restricting attention to the case of two agents and deterministic mechanisms, Drexl and Kleiner (2015) derive the optimal expected welfare maximizing DSIC and budget-balanced mechanism. Shao and Zhou (2013) do the same analysis for two agents but without requiring budget-balance. These papers illustrate difficulties in extending such analysis to more than two agents. In that sense, we provide a prior-free method of measuring welfare of mechanisms which turns out to be tractable for any number of agents.

7. Conclusion

This paper provides a novel DSIC, budget-balanced, symmetric, and ex-post individually rational mechanism to allocate a single unit of a resource. The mechanism converges to efficiency with moderately high number of agents. Further, the mechanism can be viewed as a generalization of the Green–Laffont mechanism. From a methodological standpoint, we provide several key insights on how to analyze DSIC and budget-balanced mechanisms, and propose a tractable class of mechanisms that we call ranking mechanisms.

While we carry out this analysis for allocating a single unit of resource, we feel that the ideas in this paper can be pushed in other models of mechanism design where budget-balance is a constraint. Further, an indirect implementation of our mechanism will significantly improve the practicality of our proposed mechanism.

The GL mechanism can be written as randomization over a finite (n) number of deterministic, DSIC, budget-balanced, and ex-post IR mechanisms. This leads to a simple and transparent implementation of the GL mechanism. We do not know if the r-optimal mechanism in Theorem 1 admits such an implementation.[13] It may be possible to write a two-step ranking mechanism as a convex combination of two budget-balanced mechanisms which are not DSIC, but whose convex combination becomes DSIC. For instance, we can consider two mechanisms: in mechanism 1, probability $(\pi_1 - \pi_2)$ is allocated using a Vickrey auction between two highest valued agents, but payments are adjusted to be equal to as shown in Proposition 4; in mechanism 2, probability $\ell\pi_2$ is equally shared between top ℓ-valued agents for free. While each of these mechanisms are not DSIC, their combination is DSIC. This is still not a transparent way to simplify the implementation of a two-step ranking mechanism, and a precise and neat implementation is left as a direction for future research.

From a broader perspective, our results quantify the impossibility of designing DSIC, budget-balanced, and efficient mechanisms in the single object allocation problem. It shows that even though impossibility exists, it is really thin. Thus, the possibility results with Bayesian incentive compatibility (Cramton et al., 1987) or approximate possibility results with relaxed budget-balanced constraints (Guo and Conitzer, 2009; Moulin, 2009) can also be approximately achieved with DSIC and budget-balanced mechanisms.

Appendix A. Omitted proofs

This section contains all the missing proofs. All our proofs use a fundamental result, which characterizes the entire class of satisfactory mechanism. We first prove this (Theorem 6). Once this result is proved, we finish our other proofs.

A.1. Satisfactory implementation

In this section, we provide a characterization that drives all our main results. In particular, we provide a complete characterization of allocation rules which can be satisfactorily implemented. Besides the technical aspect, there are other reasons

[13] More generally, whether every satisfactory mechanism can be written as a convex combination of a set of deterministic, DSIC, budget-balanced, and ex-post IR mechanisms remains an open question.

why such a characterization is useful: (1) it provides a recipe for carrying out such analysis of satisfactory mechanisms in other models and (2) it showcases the rich but complex class of non-ranking mechanisms that are satisfactory, thus, highlighting the salience of ranking mechanisms.

Before stating the characterization, we remind the reader about the following characterization of DSIC mechanisms by Myerson.[14]

Lemma 1 *(Myerson, 1981). A mechanism $(f, \mathbf{p})$ is DSIC if and only if*

- *(**Monotonicity of** f) for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$ with $v_i > v_i'$, we have*

$$f_i(v_i, v_{-i}) \geq f_i(v_i', v_{-i}).$$

- *(**Revenue equivalence**) for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i \in V$, we have*

$$p_i(v_i, v_{-i}) = p_i(0, v_{-i}) + v_i f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i.$$

For any mechanism $M \equiv (f, \mathbf{p})$, we define $\mathcal{U}_i^M(\mathbf{v})$ as the net utility of agent i at valuation profile $\mathbf{v}$:

$$\mathcal{U}_i^M(\mathbf{v}) = v_i f_i(\mathbf{v}) - p_i(\mathbf{v}).$$

A consequence of the Myersonian characterization of DSIC is the following characterization of DSIC and budget-balanced mechanisms. The proof is a direct application of Lemma 1, and is skipped.

Proposition 7. *A mechanism $M \equiv (f, \mathbf{p})$ is DSIC and budget-balanced if and only if*

1. *for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$ with $v_i > v_i'$ we have*

$$f_i(v_i, v_{-i}) \geq f_i(v_i', v_{-i}).$$

2. *for every $i \in N$, for every $v_{-i} \in V^{n-1}$, for every $v_i \in V$, we have*

$$\mathcal{U}_i^M(v_i, v_{-i}) = \mathcal{U}_i^M(0, v_{-i}) + \int_0^{v_i} f_i(x_i, v_{-i}) dx_i.$$

3. *for every $\mathbf{v} \equiv (v_1, \ldots, v_n) \in V^n$,*

$$\sum_{i \in N} \mathcal{U}_i^M(0, v_{-i}) = \sum_{i \in N} \Big[v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i \Big].$$

Our main characterization, like Myerson's characterization, provides a way to separate out the allocation rule and the payment rule in a satisfactory mechanism. While Myerson does not impose budget-balance, our result shows that this separation continues to hold even if we impose budget-balance.

Fix an allocation rule f. If f is monotone (in the sense of Lemma 1), then we can immediately define a payment scheme $\mathbf{p}$ that makes $(f, \mathbf{p})$ DSIC as follows: for every $i \in N$ and for every $\mathbf{v}$, set

$$p_i(\mathbf{v}) = v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i.$$

Note that $p_i(0, v_{-i}) = 0$ for all i and for all v_{-i} in this mechanism. We call a mechanism defined from such a payment scheme as the **elementary mechanism** corresponding to a monotone f. It can be easily verified that if f is the efficient allocation rule, then the corresponding elementary mechanism is the Vickrey auction.

For every valuation profile $\mathbf{v}$, define for every $i \in N$, the payment of agent i in the elementary mechanism corresponding to a monotone f as:

[14] The characterization in Myerson is for Bayesian incentive compatible mechanisms. However, it is straightforward to extend it to DSIC mechanisms.

$$R_i^f(\mathbf{v}) := v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i})dx_i.$$

Then,

$$R^f(\mathbf{v}) := \sum_{i \in N} R_i^f(\mathbf{v}),$$

denotes the total revenue collected at valuation profile $\mathbf{v}$ in the elementary mechanism corresponding to f.

We will provide necessary and sufficient conditions on f for it to be satisfactorily implementable. These conditions are given in terms of revenue collected from the elementary mechanism corresponding to f at various valuation profiles.

At any valuation profile $\mathbf{v}$, define $N^0_{\mathbf{v}} := \{i \in N : v_i = 0\}$. Given any valuation profile $\mathbf{v}$, for any $T \subseteq N$, we denote by $(0_T, v_{-T})$ the valuation profile where all the agents in T have zero valuation and each agent $i \notin T$ has valuation $v_i > 0$.

Definition 9. An allocation rule f is **residually balanced** if for every $\mathbf{v}$ such that $N^0_{\mathbf{v}} = \emptyset$, we have

$$\sum_{T \subseteq N} (-1)^{|T|} R^f(0_T, v_{-T}) = 0. \tag{1}$$

Residual balancedness is a technical combinatorial condition on an allocation rule. We show that for a symmetric and monotone allocation rule residual balancedness is necessary and sufficient for satisfactory implementability.

Theorem 6. *A symmetric allocation rule f is satisfactorily implementable if and only if it is (a) monotone and (b) residually balanced.*

Further, if f is satisfactorily implementable, then there is a unique $\mathbf{p}$ such that $(f, \mathbf{p})$ is a satisfactory mechanism. Such a unique $\mathbf{p}$ is defined as follows: for all $\mathbf{v} \in V^n$, for all $i \in N$,

$$p_i(\mathbf{v}) = -\frac{1}{|N^0_{\mathbf{v}}|} \sum_{T \subseteq N : N^0_{\mathbf{v}} \subseteq T} \frac{(-1)^{|T \setminus N^0_{\mathbf{v}}|}}{C(|T|, |N^0_{\mathbf{v}}|)} R^f(0_T, v_{-T}) \qquad \text{if } i \in N^0_{\mathbf{v}}$$

$$p_i(\mathbf{v}) = R_i^f(\mathbf{v}) - \frac{1}{|N^0_{\mathbf{v}}| + 1} \sum_{T \subseteq N : (N^0_{\mathbf{v}} \cup \{i\}) \subseteq T} \frac{(-1)^{|T \setminus N^0_{\mathbf{v}}| - 1}}{C(|T|, (|N^0_{\mathbf{v}}| + 1))} R^f(0_T, v_{-T}) \qquad \text{if } i \notin N^0_{\mathbf{v}}$$

The condition in Theorem 6 looks very similar to the cubical array lemma in Walker (1980). While the cubical array lemma applies to only efficient allocation rule, our characterization is for *any* allocation rule. Theorem 2 in Yenmez (2015) characterizes ex-post incentive compatible and budget-balanced mechanisms.[15] His characterization is a characterization of DSIC and budget-balanced *mechanisms*, and hence, still uses transfers. However, because of revenue equivalence, we can still use his characterization to derive a result similar to ours. A difference between our result and Yenmez (2015) is that we use symmetry, and hence, we construct symmetric payment rules for our allocation rules, but Yenmez (2015) does not impose symmetry. Hence, we cannot directly use the result in Yenmez (2015). For these reasons, we give below an independent proof.

Proof of Theorem 6. Suppose f is a symmetric allocation rule which is satisfactorily implementable. This implies that there exists a symmetric $\mathbf{p}$ such that the mechanism $M \equiv (f, \mathbf{p})$ is satisfactory. By Proposition 7, f is monotone. The remainder of the claims we do in steps.

Step 1. In this step, we show that for every $\mathbf{v} \in V^n$ such that $N^0_{\mathbf{v}} \neq \emptyset$, we have for every $i \in N^0_{\mathbf{v}}$,

$$\mathcal{U}_i^M(\mathbf{v}) = \frac{1}{|N^0_{\mathbf{v}}|} \sum_{T \subseteq N : N^0_{\mathbf{v}} \subseteq T} \frac{(-1)^{|T \setminus N^0_{\mathbf{v}}|}}{C(|T|, |N^0_{\mathbf{v}}|)} R^f(0_T, v_{-T}).$$

We show this using induction. If $|N^0_{\mathbf{v}}| = n$, then budget-balance implies that $\sum_{i \in N} \mathcal{U}_i^M(\mathbf{v}) = 0$. Symmetry implies that $\mathcal{U}_j^M(\mathbf{v}) = \mathcal{U}_k^M(\mathbf{v})$ for all $j, k \in N$ at this valuation profile. Hence, $\mathcal{U}_i^M(\mathbf{v}) = 0$ for all $i \in N$. Since $\mathbf{v} \equiv 0_N$, we have $R^f(\mathbf{v}) = 0$. Hence, the claim is true for $N^0 = N$.

Suppose the claim is true for all valuation profiles $\bar{\mathbf{v}}$ such that $|N^0_{\bar{\mathbf{v}}}| > |N^0_{\mathbf{v}}|$. Let $K \equiv N^0_{\mathbf{v}}$. Since M is DSIC and budget-balanced, by Proposition 7, we get

[15] His solution concept is ex-post incentive compatibility because he looks at a setting that can potentially allow for interdependent valuations.

$$\begin{aligned}
R^f(\mathbf{v}) &= \sum_{i\in N}\mathcal{U}_i^M(0,v_{-i}) = \sum_{i\in K}\mathcal{U}_i^M(0,v_{-i}) + \sum_{i\notin K}\mathcal{U}_i^M(0,v_{-i})\\
&= \sum_{i\in K}\mathcal{U}_i^M(0_K,v_{-K}) + \sum_{i\notin K}\mathcal{U}_i^M(0_{K\cup\{i\}},v_{-(K\cup\{i\})})\\
&= |K|\mathcal{U}_j^M(0_K,v_{-K}) + \sum_{i\notin K}\mathcal{U}_i^M(0_{K\cup\{i\}},v_{-(K\cup\{i\})}) \qquad \text{(where } j \text{ is some agent in } K\text{)}\\
&= |K|\mathcal{U}_j^M(0_K,v_{-K}) + \frac{1}{|K|+1}\sum_{i\notin K}\sum_{T\subseteq N:(K\cup\{i\})\subseteq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,(|K|+1))}R^f(0_T,v_{-T}),
\end{aligned}$$

where the third equality followed from symmetry and the final equality followed from the induction hypothesis. The summation in the last line of the above sequence of expressions can be simplified as follows:

$$\begin{aligned}
&\sum_{i\notin K}\sum_{T\subseteq N:(K\cup\{i\})\subseteq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,(|K|+1))}R^f(0_T,v_{-T})\\
&= \sum_{T\subseteq N:K\subsetneq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,(|K|+1))}(|T\setminus K|)\,R^f(0_T,v_{-T})\\
&= \sum_{T\subseteq N:K\subsetneq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,|K|)}(|K|+1)\,R^f(0_T,v_{-T}).
\end{aligned}$$

To understand why the first equality works, note that for every $T\subseteq N$ such that $K\subseteq T$, the summation will come for all $i\in T\setminus K$. Hence, it will appear $(|T\setminus K|)$ times.

Using the above equations in the earlier expression, we get that for all $j\in K$,

$$\begin{aligned}
\mathcal{U}_j^M(0_K,v_{-K}) &= \frac{1}{|K|}R^f(\mathbf{v}) + \frac{1}{|K|}\sum_{T\subseteq N:K\subsetneq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}\,R^f(0_T,v_{-T})\\
&= \frac{1}{|K|}R^f(0_K,v_{-K}) + \frac{1}{|K|}\sum_{T\subseteq N:K\subsetneq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}\,R^f(0_T,v_{-T})\\
&= \frac{1}{|K|}\sum_{T\subseteq N:K\subseteq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}\,R^f(0_T,v_{-T})
\end{aligned}$$

This proves the claim.

Step 2. Now consider any valuation profile $\mathbf{v}$. By Proposition 7, we see that for every agent $i\in N$,

$$p_i(\mathbf{v}) = R_i^f(\mathbf{v}) - \mathcal{U}_i^M(0,v_{-i}).$$

Using Step 1 in this equation gives us the desired expression for $p_i(\mathbf{v})$.

Step 3. Finally, we show that f is residually balanced. Consider any type profile $\mathbf{v}$ such that $N_{\mathbf{v}}^0=\emptyset$. Then, using Step 2, for every $i\in N$,

$$p_i(\mathbf{v}) = R_i^f(\mathbf{v}) - \sum_{T\subseteq N:i\in T}\frac{(-1)^{|T|-1}}{|T|}R^f(0_T,v_{-T}).$$

Hence, we get

$$\begin{aligned}
0 = \sum_{i\in N}p_i(\mathbf{v}) &= R^f(\mathbf{v}) + \sum_{i\in N}\sum_{T\subseteq N:i\in T}\frac{(-1)^{|T|}}{|T|}R^f(0_T,v_{-T})\\
&= R^f(\mathbf{v}) + \sum_{T\subseteq N:T\neq\emptyset}(-1)^{|T|}R^f(0_T,v_{-T})\\
&= \sum_{T\subseteq N}(-1)^{|T|}R^f(0_T,v_{-T}).
\end{aligned}$$

This shows that f is residually balanced. This concludes one direction of our proof.

For the other direction, suppose f is a symmetric allocation rule that is monotone and residually balanced. Consider $\mathbf{p}$ defined in the statement of this theorem. Clearly, $\mathbf{p}$ is symmetric since f is symmetric. Hence, $M \equiv (f, \mathbf{p})$ is a symmetric mechanism. Further, for every agent $i \in N$ and every valuation profile $\mathbf{v}$, we get

$$\mathcal{U}_i^M(v_i, v_{-i}) = v_i f_i(v_i, v_{-i}) - R_i^f(v_i, v_{-i}) + \mathcal{U}_i^M(0, v_{-i}),$$

where we have used the expression for $p_i(\mathbf{v})$ to substitute it with $R_i^f(\mathbf{v}) - \mathcal{U}_i^M(0, v_{-i})$ in the above expression. This gives us

$$\mathcal{U}_i^M(v_i, v_{-i}) = \mathcal{U}_i^M(0, v_{-i}) + \int_0^{v_i} f_i(x_i, v_{-i}) dx_i.$$

This along with the monotonicity of f implies M is DSIC (Proposition 7).

Finally, we show that M is budget-balanced. To do so, consider a valuation profile $\mathbf{v}$. We consider two cases.

Case 1. $N_{\mathbf{v}}^0 \neq \emptyset$. Let $K \equiv N_{\mathbf{v}}^0$. Now,

$$\begin{aligned}
\sum_{i\in N} p_i(\mathbf{v}) &= \sum_{i\in K} p_i(\mathbf{v}) + \sum_{i\notin K} p_i(\mathbf{v}) \\
&= \sum_{i\in K}\left[R_i^f(\mathbf{v}) - \frac{1}{|K|}\sum_{T\subseteq N: K\subseteq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}R^f(0_T, v_{-T})\right] \\
&+ \sum_{i\notin K}\left[R_i^f(\mathbf{v}) - \frac{1}{|K|+1}\sum_{T\subseteq N:(K\cup\{i\})\subseteq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,(|K|+1))}R^f(0_T, v_{-T})\right] \\
&= R^f(\mathbf{v}) - \sum_{T\subseteq N: K\subseteq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}R^f(0_T, v_{-T}) \\
&- \sum_{i\notin K}\left[\frac{1}{|K|+1}\sum_{T\subseteq N:(K\cup\{i\})\subseteq T}\frac{(-1)^{|T\setminus K|-1}}{C(|T|,(|K|+1))}R^f(0_T, v_{-T})\right] \\
&= R^f(\mathbf{v}) - \sum_{T\subseteq N: K\subseteq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}R^f(0_T, v_{-T}) \\
&+ \left[\sum_{T\subseteq N: K\subsetneq T}\frac{(-1)^{|T\setminus K|}}{C(|T|,|K|)}R^f(0_T, v_{-T})\right] \\
&= R^f(\mathbf{v}) - R^f(0_K, v_{-K}) \\
&= 0.
\end{aligned}$$

Note that budget-balance followed without any extra conditions in this case.

Case 2. $N_{\mathbf{v}}^0 = \emptyset$. In that case,

$$\begin{aligned}
\sum_{i\in N} p_i(\mathbf{v}) &= R^f(\mathbf{v}) + \sum_{i\in N}\sum_{T\subseteq N: i\in T}\frac{(-1)^{|T|}}{|T|}R^f(0_T, v_{-T}) \\
&= R^f(\mathbf{v}) + \sum_{T\subseteq N: T\neq\emptyset}(-1)^{|T|}\, R^f(0_T, v_{-T}) \\
&= \sum_{T\subseteq N}(-1)^{|T|}\, R^f(0_T, v_{-T}) \\
&= 0,
\end{aligned}$$

where the last equality follows from the fact that f is residually balanced.

This completes the proof. $\square$

A.2. Proof of results in the paper

Notations. We will need some extra notations. At every valuation profile $\mathbf{v}$ and for every $k \in N$, we denote by $v_{(k)}$ the valuation of every agent in $\mathbf{v}[k]$. Note that for some $k \in N$, it is possible that $\mathbf{v}[k] = \emptyset$, in which case $v_{(k)}$ is defined to be 0. For any $j \in N$, let the cardinality of the set $\cup_{h=1}^{j} \mathbf{v}[h]$ be L_j.

We illustrate these notations with an example. Suppose $N = \{1, \ldots, 8\}$. Consider a valuation profile $\mathbf{v}$ such that $\mathbf{v}[1] = \{1, 2\}$, $\mathbf{v}[2] = \{3, 4, 5, 6\}$, and $\mathbf{v}[3] = \{7, 8\}$. Then, $L_1 = 2, L_2 = 6, L_3 = 8$. According to a ranking allocation rule with probabilities $(\pi_1, \ldots, \pi_8)$, agents 1 and 2 share $\pi_1 + \pi_2$ equally, agents 3, 4, 5, 6 share $\pi_3 + \pi_4 + \pi_5 + \pi_6$ equally and agents 7 and 8 share $\pi_7 + \pi_8$ equally. In other words, for every $j \in N$, agents in $\mathbf{v}[j]$ share equally the probabilities

$$\pi_{L_{j-1}+1} + \ldots + \pi_{L_j},$$

where $L_0 \equiv 0$.

We begin by a lemma, which will be useful to all our proofs.

Lemma 2. *Suppose f is a ranking allocation rule. Then, R^f is continuous.*

Proof. The proof is a straightforward consequence of the following observation. Fix any positive integer $K \leq n$. Consider the allocation rule f^K defined as follows: at any valuation profile, top K valuation agents receive a probability equal to $\hat{\pi}$ such that $K\hat{\pi} \leq 1$ – note that this inequality can be strict. We call such allocation rules **equal-share** allocation rules. Such an allocation rule is clearly monotone and hence implementable – indeed, the elementary mechanism corresponding to this will have each of the top K valuation agent paying the $(K+1)$-st highest valuation times $\hat{\pi}$. Hence, $R^{f^K}(\mathbf{v}) = K\hat{\pi} v_{(K+1)}$, where $v_{(K+1)}$ is the $(K+1)$-st highest valuation, and we follow the convention that $v_{(n+1)} = 0$. Clearly, R^{f^K} is continuous in $\mathbf{v}$. Hence, every equal share rule satisfies the continuity property.

Now, any ranking allocation rule can be expressed as sum of such equal share rules. To see this, take a ranking allocation rule $(\pi_1, \ldots, \pi_n)$. This can be expressed as a sum of n equal share allocation rules: (1) allocation rule which assigns π_n to each of the agents; (2) allocation rule which assigns $(\pi_{(n-1)} - \pi_n)$ to top $(n-1)$ valuation agents; (3) allocation rule which assigns $(\pi_{(n-2)} - \pi_{(n-1)})$ to top $(n-2)$ valuation agents and so on till; (n) allocation rule which assigns $(\pi_1 - \pi_2)$ to the top valuation agent. By the argument in our previous paragraph about continuity of equal share rules, the continuity of any ranking allocation rule follows. $\square$

A.3. Proof of Proposition 1

In this section, we give a proof of Proposition 1. We extensively use Theorem 6 to prove our result. Before starting our proofs, we explicitly compute the R^f values for any ranking allocation rule f. A valuation profile $\mathbf{v}$ is called **0-generic** if for all $i \neq j$ with $v_i = v_j$ we have $v_i = v_j = 0$.

We start off with the following claim.

Lemma 3. *Suppose f is a ranking allocation rule with allocation probabilities $\pi \equiv (\pi_1, \ldots, \pi_n)$. Then, for every 0-generic valuation profile $\mathbf{v}$, we have*

$$R^f(\mathbf{v}) = \sum_{j=1}^{n-1} j v_{(j+1)} (\pi_j - \pi_{j+1}),$$

where $v_{(k)} = 0$ if $\mathbf{v}[k] = \emptyset$ for any k.

Proof. Choose a 0-generic valuation profile $\mathbf{v}$. Consider agent $i \in N$ with $v_i > 0$. Since $\mathbf{v}$ is a 0-generic valuation profile, $\{i\} = \mathbf{v}[j]$ for some j. If $j = n$, then $v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i = 0$. So, consider $j < n$. As a result

$$\begin{aligned} v_i f_i(\mathbf{v}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i &= \pi_j v_{(j)} - \int_0^{v_{(j)}} f_i(x_i, v_{-i}) dx_i \\ &= \pi_j v_{(j)} - \sum_{h=j}^{n} \pi_h (v_{(h)} - v_{(h+1)}) \qquad \text{(Note: } v_{(n+1)} \equiv 0.\text{)} \\ &= \sum_{h=j+1}^{n} v_{(h)} (\pi_{h-1} - \pi_h). \end{aligned}$$

This implies that $R^f(\mathbf{v}) = \sum_{j=1}^{n-1}\sum_{h=j+1}^{n} v_{(h)}(\pi_{h-1}-\pi_h) = \sum_{j=1}^{n-1} j v_{(j+1)}(\pi_j - \pi_{j+1})$. $\square$

Using Lemma 3, we will now give a proof of Proposition 1.

Proof of Proposition 1. Let f be a ranking allocation rule with allocation probabilities $(\pi_1,\ldots,\pi_n)$. Note that f is monotone in the sense of Myerson. By Theorem 6, we know that f is satisfactorily implementable if and only if for every $\mathbf{v}$ with $v_1 \geq v_2 \geq \ldots \geq v_n > 0$, we have

$$\sum_{T\subseteq N}(-1)^{|T|}R^f(0_T, v_{-T}) = \sum_{k=0}^{n}\sum_{T\subseteq N:|T|=n-k}(-1)^{n-k}R^f(0_T, v_{-T}) = 0.$$

Since R^f is continuous (Lemma 2), it is enough to show the above equality for 0-generic valuation profiles. In other words, continuity of R^f implies that f is satisfactorily implementable if and only if for every $\mathbf{v}$ with $v_1 > v_2 > \ldots > v_n > 0$, we have

$$\sum_{T\subseteq N}(-1)^{|T|}R^f(0_T, v_{-T}) = \sum_{k=0}^{n}\sum_{T\subseteq N:|T|=k}(-1)^{k}R^f(0_T, v_{-T}) = 0.$$

Note that for every $T\subseteq N$, the profile $(0_T, v_{-T})$ is a 0-generic valuation profile. We can divide this sum into two parts.

$$\sum_{T\subseteq N}(-1)^{|T|}R^f(0_T, v_{-T}) = \sum_{T\subseteq N:n\in T}(-1)^{|T|}R^f(0_T, v_{-T}) + \sum_{T\subseteq N:n\notin T}(-1)^{|T|}R^f(0_T, v_{-T})$$

Hence, we can write the residual balancedness condition as

$$\sum_{T\subseteq N}(-1)^{|T|}R^f(0_T, v_{-T}) = \sum_{T\subseteq N:n\notin T}(-1)^{|T|}\Big[R^f(0_T, v_{-T}) - R^f(0_{T\cup\{n\}}, v_{-(T\cup\{n\})})\Big] = 0.$$

Now, fix a $T\subseteq N$ with $n\notin T$ and $|T| = n-k$. Since $\mathbf{v}$ is a 0-generic valuation profile, the rank of agent n in $(0_T, v_{-T})$ is k. Without loss of generality, we denote $(0_T, v_{-T}) \equiv \mathbf{v}'$. Note that $v'_{(k)} = v_n$. Using Lemma 3,

$$R^f(0_T, v_{-T}) = \sum_{j=1}^{k-1} j v'_{(j+1)}(\pi_j - \pi_{j+1})$$

and

$$R^f(0_{(T\cup\{n\})}, v_{-(T\cup\{n\})}) = \sum_{j=1}^{k-2} j v'_{(j+1)}(\pi_j - \pi_{j+1}).$$

Hence, we can write

$$R^f(0_T, v_{-T}) - R^f(0_{T\cup\{n\}}, v_{-(T\cup\{n\})}) = (k-1)v'_{(k)}(\pi_{k-1}-\pi_k) = (k-1)v_n(\pi_{k-1}-\pi_k),$$

where the last equality follows because $v'_{(k)} = v_n$. Note that the RHS only depends on the size of T but not on which elements are present in T. As a result, we can write the residual balancedness condition as

$$\begin{aligned} 0 = \sum_{T\subseteq N}(-1)^{|T|}R^f(0_T, v_{-T}) &= \sum_{T\subseteq N:n\notin T}(-1)^{|T|}\Big[R^f(0_T, v_{-T}) - R^f(0_{T\cup\{n\}}, v_{-(T\cup\{n\})})\Big] \\ &= \sum_{k=1}^{n}\sum_{T\subseteq N:n\notin T,|T|=n-k}(-1)^{n-k}(k-1)v_n(\pi_{k-1}-\pi_k) \\ &= \sum_{k=2}^{n}(-1)^{n-k}C(n-1,k-1)(k-1)(\pi_{k-1}-\pi_k)v_n. \end{aligned}$$

The last inequality follows because we can form a subset of size $n-k$ from $n-1$ elements in $C(n-1,k-1)$ ways. Now, we simplify this expression to get our desired result. Since $v_n > 0$, residual balancedness is equivalent to:

$$\begin{aligned} 0 &= \sum_{k=2}^{n}(-1)^{n-k}C(n-1,k-1)(k-1)(\pi_{k-1}-\pi_k) \\ &= \sum_{k=2}^{n}(-1)^{n-k}C(n-1,k-1)(k-1)\pi_{k-1} - \sum_{k=2}^{n}(-1)^{n-k}C(n-1,k-1)(k-1)\pi_k \end{aligned}$$

$$= -\sum_{\ell=1}^{n-1}(-1)^{n-\ell}C(n-1,\ell)\ell\pi_\ell - \sum_{\ell=1}^{n}(-1)^{n-\ell}C(n-1,\ell-1)(\ell-1)\pi_\ell$$

$$= -\sum_{\ell=1}^{n-1}(-1)^{n-\ell}\pi_\ell\big[\ell C(n-1,\ell)+(\ell-1)C(n-1,\ell-1)\big] - (-1)^0(n-1)C(n-1,n-1)\pi_n$$

$$= -\sum_{\ell=1}^{n-1}(-1)^{n-\ell}(n-1)C(n-1,\ell-1)\pi_\ell - (-1)^0(n-1)C(n-1,n-1)\pi_n$$

(Here, we used the fact that $\ell C(n-1,\ell)+(\ell-1)C(n-1,\ell-1)=(n-1)C(n-1,\ell-1)$.)

$$= -\sum_{\ell=1}^{n}(-1)^{n-\ell}(n-1)C(n-1,\ell-1)\pi_\ell$$

Since $n>1$, we get that residual balancedness is equivalent to

$$0=\sum_{\ell=1}^{n}(-1)^{n-\ell}C(n-1,\ell-1)\pi_\ell.$$

This can be equivalently written as

$$0=\sum_{\ell=1}^{n}(-1)^{\ell}C(n-1,\ell-1)\pi_\ell,$$

which is the desired claim. □

A.4. Proofs of Theorem 1 and Theorem 3

In this section, we give a proof of Theorems 1 and 3. We start by characterizing the two-step ranking allocation rules that can be satisfactorily implemented.

Proposition 8. *A two-step ranking allocation rule is satisfactorily implementable if and only if $2 \le \ell \le n-1$, ℓ is even, and*

$$\pi_1 = \frac{C(n-2,\ell-1)+1}{C(n-2,\ell-1)+\ell}.$$

Proof. In this and subsequent proofs, we use the following combinatorial fact.

Fact 1. *For any $r \in \{0,\dots,n-1\}$,*

$$\sum_{j=0}^{r}(-1)^j C(n,j) = (-1)^r C(n-1,r)$$

and

$$\sum_{j=0}^{n}(-1)^j C(n,j) = 0.$$

By Proposition 1, we know that for any two-step ranking allocation rule defined by (π_1,ℓ), satisfactorily implementability is equivalent to

$$-\pi_1+\sum_{k=2}^{\ell}(-1)^k C(n-1,k-1)\pi_2 = 0. \tag{2}$$

This immediately implies that $\ell \neq 1$. Further, if $\ell = n$, then we must have $\pi_1 = \sum_{k=2}^{n}(-1)^k C(n-1,k-1)\pi_2 = \pi_2$. But, by definition of a two-step allocation rule $\pi_1 > \pi_2$. So, we have $1<\ell<n$.

Now, using Fact 1,

$$\begin{aligned}\sum_{k=2}^{\ell}(-1)^k C(n-1,k-1) &= -\sum_{k=1}^{\ell-1}(-1)^k C(n-1,k)\\ &= 1-\Big[\sum_{k=0}^{\ell-1}(-1)^k C(n-1,k)\Big]\\ &= 1-(-1)^{\ell-1}C(n-2,\ell-1)\\ &= 1+(-1)^{\ell}C(n-2,\ell-1).\end{aligned}$$

Using this and the fact that $\pi_2=\frac{1}{\ell-1}(1-\pi_1)$, we simplify Equation (2) as

$$-\pi_1+\frac{1}{\ell-1}(1-\pi_1)\Big(1+(-1)^{\ell}C(n-2,\ell-1)\Big)=0.$$

For this to hold, we must have ℓ even and

$$\pi_1=\frac{C(n-2,\ell-1)+1}{C(n-2,\ell-1)+\ell}. \qquad \square$$

We now provide a proof of Theorem 3 first. We will use Theorem 3 to derive the proof of Theorem 1.

Proof of Theorem 3. We do the proof in several steps.

Step 1 – The Primal Problem. In this step, we formulate the problem of finding a rew-optimal allocation rule as the following linear program.

$$\begin{aligned} &\max_{(\pi_1,\ldots,\pi_n)} \sum_{i=1}^{n} v_i\pi_i\\ &\text{s.t.} \qquad \textbf{(LP-RANK2)}\\ &\pi_{i+1}-\pi_i\le 0 \qquad \forall\, i\in\{1,\ldots,n-1\}\\ &\sum_{i=1}^{n}(-1)^i C(n-1,i-1)\pi_i=0\\ &\sum_{i=1}^{n}\pi_i\le 1\\ &\pi_i\ge 0 \qquad \forall\, i\in\{1,\ldots,n\}.\end{aligned}$$

By Proposition 1, a feasible solution to the linear program **(LP-RANK2)** is a satisfactorily implementable ranking allocation rule.

Step 2 – The Dual Problem. We first consider the dual of **(LP-RANK2)** and construct a dual feasible solution. For formulating the dual, we associate a variable θ_i for each of the constraint in the first set of constraints corresponding to $i\in\{1,\ldots,n-1\}$. We also associate variables y and z for the second and third constraints respectively.

This leads us to the dual of the linear program **(LP-RANK2)**.

$$\begin{aligned} &\min_{(y,z,(\theta_1,\ldots,\theta_{n-1}))} z\\ &\text{s.t.} \qquad \textbf{(DP-RANK2)}\\ &-\theta_1-y+z\ge v_1\\ &\theta_{i-1}-\theta_i+(-1)^i C(n-1,i-1)y+z\ge v_i \qquad \forall\, i\in\{2,\ldots,n-1\}\\ &\theta_{n-1}+(-1)^n y+z\ge v_n\\ &\theta_i\ge 0 \qquad \forall\, i\in\{1,\ldots,n-1\}\\ &z\ge 0.\end{aligned}$$

We construct a dual feasible solution as follows. Set $\theta_1 = 0$ and we will choose y and z such that $z - y = \nu_1$. This will imply that the first constraint is automatically satisfied. The rest of the constraints are satisfied by successively computing θ_i for $i \in \{2, \dots, n-1\}$. First, we set

$$\theta_2 = \theta_1 + (-1)^2 C(n-1,1)y + z - \nu_2 = (-1)^2 C(n-1,1)y + z - \nu_2.$$

Then,

$$\theta_3 = \theta_2 + (-1)^3 C(n-1,2)y + z - \nu_3 = \Big((-1)^2 C(n-1,1) + (-1)^3 C(n-1,2)\Big)y + 2z - \nu_2 - \nu_3.$$

Continuing in this manner, we have for all $i \in \{2, \dots, n-1\}$,

$$\begin{aligned}
\theta_i &= \Big(\sum_{j=1}^{i-1} (-1)^{j+1} C(n-1,j)\Big)y + (i-1)z - \sum_{j=2}^{i} \nu_j \\
&= (i-1)z - \sum_{j=2}^{i} \nu_j - \Big(\sum_{j=1}^{i-1} (-1)^{j} C(n-1,j)\Big)y \\
&= (i-1)z - \sum_{j=2}^{i} \nu_j - \Big((-1)^{i-1} C(n-2,i-1) - 1\Big)y \qquad \text{(using Fact 1)} \\
&= (i-1)z - \sum_{j=2}^{i} \nu_j - \Big((-1)^{i-1} C(n-2,i-1) - 1\Big)z \\
&\quad + \nu_1\Big((-1)^{i-1} C(n-2,i-1) - 1\Big) \qquad \text{(using } y = z - \nu_1\text{)} \\
&= \nu_1\Big((-1)^{i-1} C(n-2,i-1) - 1\Big) - z\Big((-1)^{i-1} C(n-2,i-1) - i\Big) - \sum_{j=2}^{i} \nu_j.
\end{aligned}$$

This choice of θ_i ensures that the second set of inequalities in **DP-RANK2** are satisfied. However, we need to make sure that (a) θ_is are non-negative and (b) the last inequality is satisfied. These are ensured by choosing y and z appropriately.

For every $i \in \{2, \dots, n-1\}$, let

$$H(n,i) := (-1)^{i-1} C(n-2,i-1).$$

First, for non-negativity of θ_i, we will choose z appropriately. Note that $\nu_i \geq 0$ for all i. Further, $\theta_i \geq 0$ if and only if

$$\nu_1\Big(H(n,i) - 1\Big) - z\Big(H(n,i) - i\Big) - \sum_{j=2}^{i} \nu_j \geq 0. \tag{3}$$

We consider two cases.

Case a. If i is even, we have $H(n,i) = -C(n-2,i-1) < 0$. Simplifying, we get

$$\begin{aligned}
z &\geq \nu_1 \frac{\Big(C(n-2,i-1) + 1\Big)}{\Big(C(n-2,i-1) + i\Big)} + \frac{1}{\Big(C(n-2,i-1) + i\Big)} \sum_{j=2}^{i} \nu_j \\
&= \nu_1\left(1 - \frac{(i-1)}{\Big(C(n-2,i-1) + i\Big)}\right) + \frac{1}{\Big(C(n-2,i-1) + i\Big)} \sum_{j=2}^{i} \nu_j.
\end{aligned} \tag{4}$$

Note that if $i = 2$, we need

$$z \geq \nu_1(1 - \frac{1}{n}) + \frac{1}{n}\nu_2 \tag{5}$$

Now, choose ℓ as follows:

$$\ell \in \arg\max_{n-1 \geq i \geq 2,\ i \text{ even}} \left[\nu_1\left(1 - \frac{(i-1)}{\Big(C(n-2,i-1) + i\Big)}\right) + \frac{1}{\Big(C(n-2,i-1) + i\Big)} \sum_{j=2}^{i} \nu_j\right]. \tag{6}$$

Inequality (4) can be satisfied by choosing $z = z^*$, where

$$z^* := \left[\nu_1 \left(1 - \frac{(\ell - 1)}{\left(C(n-2, \ell-1) + \ell\right)} \right) + \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} \sum_{j=2}^{\ell} \nu_j \right]. \tag{7}$$

As argued earlier, $z^* \geq \nu_1(1 - \frac{1}{n}) + \frac{1}{n}\nu_2$.

Case B. If i is odd, then $H(n,i) = C(n-2, i-1)$. We show below that the choice of $z = z^*$ (defined in Equation (7)) works in this case too. If $i = n-1$, then Inequality (3) reduces to $z(n-2) - \sum_{j=2}^{n-1} \nu_j \geq 0$. For $z = z^*$, we know that $z^* \geq (1 - \frac{1}{n})\nu_1 + \frac{1}{n}\nu_2$ – by Inequality (5). Hence, $z^*(n-2) \geq \nu_1 \frac{(n-2)(n-1)}{n} + \nu_2 \frac{(n-2)}{n} \geq \nu_2 \frac{(n-2)}{n}\left((n-1)+1\right) = (n-2)\nu_2 \geq \sum_{j=2}^{n-1} \nu_j$, where the last two inequality follows from the fact $\nu_1 \geq \nu_2$ and $\nu_2 \geq \nu_j$ for all $j \geq 2$.

Hence, we assume $i < n-1$. In that case $H(n,i) \geq i$. If $H(n,i) - i = 0$, then the desired Inequality (3) is satisfied since $\nu_1 \geq \nu_j$ for all $j \neq 1$ gives us:

$$\nu_1\left(H(n,i) - 1\right) - \sum_{j=2}^{i} \nu_j \geq \nu_1(i-1) - \sum_{j=2}^{i} \nu_j \geq \nu_1(i-1) - \nu_1(i-1) = 0.$$

Assume that $H(n,i) > i$. Then, Inequality (3) holds if

$$\begin{aligned}
z &\leq \nu_1 \frac{\left(C(n-2, i-1) - 1\right)}{\left(C(n-2, i-1) - i\right)} - \frac{1}{\left(C(n-2, i-1) - i\right)} \sum_{j=2}^{i} \nu_j \\
&= \nu_1 \left(1 + \frac{(i-1)}{\left(C(n-2, i-1) - i\right)} \right) - \frac{1}{\left(C(n-2, i-1) - i\right)} \sum_{j=2}^{i} \nu_j.
\end{aligned}$$

We argue that this is satisfied if $z = z^*$. To see this, note that

$$\begin{aligned}
z^* &= \left[\nu_1 \left(1 - \frac{(\ell - 1)}{\left(C(n-2, \ell-1) + \ell\right)} \right) + \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} \sum_{j=2}^{\ell} \nu_j \right] \\
&\leq \nu_1 \left(1 - \frac{(\ell - 1)}{\left(C(n-2, \ell-1) + \ell\right)} \right) + \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} (\ell - 1)\nu_1 \\
&= \nu_1 \\
&= \nu_1 \left(1 + \frac{(i-1)}{\left(C(n-2, i-1) - i\right)} \right) - \frac{(i-1)}{\left(C(n-2, i-1) - i\right)} \nu_1 \\
&\leq \nu_1 \left(1 + \frac{(i-1)}{\left(C(n-2, i-1) - i\right)} \right) - \frac{1}{\left(C(n-2, i-1) - i\right)} \sum_{j=2}^{i} \nu_j,
\end{aligned}$$

where we used the fact that $\nu_1 \geq \nu_j$ for all $j \neq 1$ for both the inequalities.

Hence, we choose $z = z^*$, where

$$z^* = \nu_1 \left(1 - \frac{(\ell - 1)}{\left(C(n-2, \ell-1) + \ell\right)} \right) + \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} \sum_{j=2}^{\ell} \nu_j. \tag{8}$$

This completes the proof that $\theta_i \geq 0$ for all i. Since $z^* \geq 0$, we satisfy the non-negativity constraints by this choice of z.

Let $y^* = z^* - \nu_1$. Finally, we show that the last inequality in **DP-RANK2** is satisfied. To see this, if n is odd, then the inequality reduces to

$$\theta_{n-1} - y^* + z^* = \theta_{n-1} + \nu_1 \geq \nu_n,$$

where the inequality follows since we have chosen $\theta_{n-1} \geq 0$ and $\nu_1 \geq \nu_n$.

If n is even, we need to show $\theta_{n-1} + y + z \geq \nu_n$. Using $y^* = z^* - \nu_1$, we need to show that $\theta_{n-1} + 2z^* \geq \nu_1 + \nu_n$. Note that $\theta_{n-1} = z^*(n-2) - \sum_{j=2}^{n-1} \nu_j$ by definition. Hence, we need to show that $nz^* \geq \sum_{j=1}^{n} \nu_j$. But using Inequality (5), we note that

$$nz^* \geq v_1(n-1) + v_2 = v_1 + v_2 + (n-2)v_1 \geq v_1 + v_2 + \sum_{j=3}^{n} v_j = \sum_{j=1}^{n} v_j,$$

where the last inequality follows from the fact $v_1 \geq v_j$ for all $j \neq 1$.

This completes the proof that there is a feasible solution of **(DP-RANK2)** with z^* defined by Equation (8).

Step 3 – Optimality. In this step, we construct a feasible solution of **(LP-RANK2)** by constructing the probabilities of a two-step ranking allocation rule as follows:

$$\pi_1^* = 1 - \frac{\ell - 1}{\left(C(n-2, \ell-1) + \ell\right)}$$

$$\pi_i^* = \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} \qquad \forall\, i \in \{2, \ldots, \ell\}$$

$$\pi_i^* = 0 \qquad \forall\, i \in \{\ell+1, \ldots, n\}.$$

By construction, $\sum_{j \in N} \pi_j^* = 1$ and $\pi_1^* \geq \pi_i^*$ for all $i \in \{2, \ldots, \ell\}$. By Proposition 8, $(\pi_1^*, \ldots, \pi_n^*)$ is a feasible solution of **(LP-RANK)**. Further, we see that the objective function value of **(LP-RANK)** with this feasible solution is

$$v_1\left(1 - \frac{\ell - 1}{\left(C(n-2, \ell-1) + \ell\right)}\right) + \sum_{j=2}^{\ell} v_j \frac{1}{\left(C(n-2, \ell-1) + \ell\right)} = z^*,$$

which is the objective function value of **(DP-RANK2)** for the dual feasible solution we found in Step 2. Hence, by the strong duality theorem of linear programming, $(\pi_1^*, \ldots, \pi_n^*)$ is an optimal solution of **(LP-RANK2)**. Hence, it describes a rew-optimal allocation rule. □

Proof of Theorem 1. The proof follows from Theorem 3. We think of a degenerate distribution where $v_1 = 1$ and $v_j = 0$ for all $j \neq 1$. Note that the optimal solution of **(LP-RANK2)** in that case finds an r-optimal allocation rule.

By Theorem 3, the r-optimal allocation rule is a 2-step ranking allocation rule with ℓ chosen as:

$$\ell \in \arg \max_{i: 2 \leq i \leq (n-1),\ i \text{ even}} \left[1 - \frac{(i-1)}{C(n-2, i-1) + i}\right].$$

But this is equivalent to choosing ℓ such that

$$\ell \in \arg \min_{i: 2 \leq i \leq (n-1),\ i \text{ even}} \frac{(i-1)}{C(n-2, i-1) + i}$$

This is the desired expression. □

A.5. Proof of Theorem 2

In this section, we provide a proof of individual rationality of a class of two-step ranking mechanisms. First, we remind the following elementary fact from Myerson (1981).

Fact 2. *A mechanism $(f, \mathbf{p})$ is ex-post individually rational if and only if for every $i \in N$ and for every v_{-i}, we have $p_i(0, v_{-i}) \leq 0$.*

Note that the above fact is a *necessary and sufficient* condition for IR. We now present two useful lemmas that will help us prove Theorem 2.

Lemma 4. *Suppose $(f, \mathbf{p})$ is a satisfactory mechanism, where f is a two-step ranking allocation rule defined by (π_1, ℓ). Then, for every $\mathbf{v}$ with $|N_{\mathbf{v}}^0| = n - K$, $K \leq \ell$, and $v_1 > \ldots > v_K > 0$, we have for every $i \in N_{\mathbf{v}}^0$,*

$$p_i(\mathbf{v}) = -\frac{(\pi_1 - \pi_2)}{\psi(n-K, n-2)}\left[\sum_{j=2}^{K-1} (-1)^j (j-1)!\,\psi(n-K, n-j-1) v_j + (-1)^K (K-1)!\, v_K\right], \text{ if } K \geq 2,$$

and $p_i(\mathbf{v}) = 0$ if $K \in \{0, 1\}$.

Lemma 5. *Suppose $(f, \mathbf{p})$ is a satisfactory mechanism, where f is a two-step ranking allocation rule defined by (π_1, ℓ). Then, for every $\mathbf{v}$ with $|N^0_{\mathbf{v}}| = n - K$, $K \geq \ell + 1$, and $v_1 > \ldots > v_K > 0$, we have for every $i \in N^0_{\mathbf{v}}$,*

$$p_i(\mathbf{v}) = -\frac{(\pi_1 - \pi_2)}{\psi(n-\ell, n-2)}\Big[\sum_{k=2}^{\ell-1}(-1)^k (k-1)!\psi(n-\ell, n-k-1)v_k + (-1)^\ell(\ell-1)!v_\ell\Big].$$

The proofs of Lemma 4 and Lemma 5 are tedious but a straightforward application of the revenue equivalence formula. We give their proofs in the supplementary appendix. With the help of these two lemmas, we can now present the proof of Theorem 2.

Proof of Theorem 2. Consider a two-step allocation rule (π_1, ℓ) such that $2\ell \leq n+1$. Proposition 8 characterizes the two-step allocation rules that are satisfactorily implementable. If $\mathbf{p}$ is such that $(f, \mathbf{p})$ is a satisfactory mechanism, then it is ex-post individually rational (by Fact 2) if and only if for every $i \in N$ and for every $\mathbf{v}$, we have $p_i(0, v_{-i}) \leq 0$.

Fix $i \in N$ and choose a profile $(0, v_{-i})$. By Lemma 2, R^f is continuous in $\mathbf{v}$. Hence, by the expression of $p_i(0, v_{-i})$ in Theorem 6, $p_i(0, v_{-i})$ is continuous in v_{-i}. Hence, we only consider v_{-i} such that $(0, v_{-i})$ is 0-generic. Thus, we can apply Lemmas 4 and 5 to compute $p_i(0, v_{-i})$ and show that it is non-positive.

Suppose $v_1 > v_2 > \ldots > v_K > 0$ and $v_j = 0$ for all $j > K$. By Lemmas 4 and 5, $p_i(0, v_{-i}) \leq 0$ if and only if for every $K \leq \ell$, the following summation is non-negative:

$$\Big[\sum_{j=2}^{K-1}(-1)^j (j-1)!\psi(n-K, n-j-1)v_j + (-1)^K(K-1)!v_K\Big].$$

Expanding this, we get

$$1!\psi(n-K, n-3)v_2 - 2!\psi(n-K, n-4)v_3 + \ldots + (-1)^K(K-1)!\psi(n-K, n-K-1)v_K, \tag{9}$$

where we abused notation to define $\psi(n-K, n-K-1) \equiv 1$. Note that if K is even the last term of Expression (9) is positive. In that case, it is sufficient to show that this summation is non-negative till $K-1$ (i.e., the last negative term in the expression). This idea is captured by considering the summation till $\lfloor K \rfloor_o$ (the largest odd number less than or equal to K). Hence, we need to show the following expression is non-negative:

$$\begin{aligned}
&\sum_{j=2}^{\lfloor K \rfloor_o}(-1)^j (j-1)!\psi(n-K, n-j-1)v_j \\
&= \sum_{2 \leq j \leq \lfloor K \rfloor_o : j \text{ even}} \Big[(j-1)!\psi(n-K, n-j-1)v_j - (j!)\psi(n-K, n-j-2)v_{j+1}\Big] \\
&= \sum_{2 \leq j \leq \lfloor K \rfloor_o : j \text{ even}} (j-1)!\psi(n-K, n-j-2)\Big[(n-j-1)v_j - jv_{j+1}\Big] \\
&\geq \sum_{2 \leq j \leq \lfloor K \rfloor_o : j \text{ even}} (j-1)!\psi(n-K, n-j-2)(n-2j-1)v_j.
\end{aligned}$$

Note that we are consider a 2-step allocation rule (π, ℓ) such that $2\ell \leq n+1$. Since $K \leq \ell$, for every $2 \leq j \leq \lfloor K \rfloor_o : j$ even, we have $j+1 \leq \ell$. Hence, for every $2 \leq j \leq \lfloor K \rfloor_o : j$ even, we have $2(j+1) \leq n+1$ or $n-2j-1 \geq 0$. This implies that the above expression is non-negative, which completes the proof. □

A.6. Proofs of Propositions 5 and 6

We complete the missing proofs of Section 4.1.

Proof of Proposition 5. For the case of uniform distribution with support $[0, \beta]$, $v_j - v_{j+1} = \frac{\beta}{(n+1)}$ for all $j \in \{1, \ldots, n-1\}$. As a result, $v_1 - v_j = (j-1)\frac{\beta}{(n+1)}$ for all $j \in \{2, \ldots, n\}$. Hence, the value of the maximizing expression in the statement of Theorem 3 can be simplified as follows:

$$\begin{aligned}
&v_1\Big(1 - \frac{(i-1)}{\big(C(n-2, i-1) + i\big)}\Big) + \frac{1}{\big(C(n-2, i-1) + i\big)}\sum_{j=2}^{i} v_j \\
&= v_1 - \frac{1}{\big(C(n-2, i-1) + i\big)}(v_1 - \sum_{j=2}^{i} v_j)
\end{aligned}$$

$$= \nu_1 - \frac{1}{\big(C(n-2,i-1)+i\big)}\Big(\sum_{j=2}^{i}(\nu_1-\nu_j)\Big)$$
$$= \nu_1 - \frac{1}{\big(C(n-2,i-1)+i\big)}\Big(\sum_{j=2}^{i}(j-1)\frac{\beta}{n+1}\Big)$$
$$= \beta\frac{n}{n+1} - \frac{\beta}{n+1}\frac{1}{\big(C(n-2,i-1)+i\big)}\frac{i(i-1)}{2}.$$

Hence, maximizing the above expression is equivalent to

$$\min_{2\le i\le (n-1):i \text{ even}} \frac{i(i-1)}{C(n-2,i-1)+i}.$$

Let ℓ^* be an optimal solution of the r-optimal mechanism computed in Theorem 1. Note that, by Theorem 1,

$$\ell^* \in \arg\min_{2\le i\le (n-1):i \text{ even}} \frac{(i-1)}{C(n-2,i-1)+i}. \tag{10}$$

Now, consider

$$\ell \in \arg\min_{2\le i\le (n-1):i \text{ even}} \frac{i(i-1)}{C(n-2,i-1)+i}. \tag{11}$$

We will argue that $\ell \le \ell^*$. Assume for contradiction $\ell > \ell^*$. Then, by Equation (10)

$$\frac{(\ell^*-1)}{C(n-2,\ell^*-1)+\ell^*} \le \frac{(\ell-1)}{C(n-2,\ell-1)+\ell}$$

Using the fact that $\ell > \ell^*$, we get

$$\frac{\ell^*(\ell^*-1)}{C(n-2,\ell^*-1)+\ell^*} < \frac{\ell(\ell-1)}{C(n-2,\ell-1)+\ell}$$

But this contradicts Equation (11).

This means $\ell \le \ell^*$. By Proposition 2, we know that $\ell^* \le \frac{n+1}{2}$. Hence, we have $\ell \le \frac{n+1}{2}$. Using Theorem 2, we conclude that the rew-optimal mechanism is ex-post individually rational. □

Proof of Proposition 6. Fix a distribution F. Let $\nu_1,\dots,\nu_n$ be the average order statistics corresponding to F. Since $n \ge 10$, the GL and the r-optimal mechanisms are different – in particular, the value of ℓ in the r-optimal mechanism is at least 4 (see Proposition 2).

Note that if a distribution F satisfies Assumption A, then for any $j \ge 3$, we have

$$\nu_1-\nu_j = (\nu_1-\nu_2)+(\nu_2-\nu_3)+\ldots+(\nu_{j-1}-\nu_j) \le (j-1)(\nu_1-\nu_2), \tag{12}$$

where the last inequality uses Assumption A.

The expected welfare from the r-optimal mechanism (with value of ℓ given in Theorem 1) is

$$\nu_1\Big[1-\frac{\ell-1}{C(n-2,\ell-1)+\ell}\Big]+\Big(\sum_{j=2}^{\ell}\nu_j\Big)\frac{1}{C(n-2,\ell-1)+\ell}.$$

The expected welfare from the GL mechanism is

$$\nu_1\Big[1-\frac{1}{n}\Big]+\nu_2\frac{1}{n}.$$

Difference in expected welfare is given by

$$\frac{1}{n}(\nu_1-\nu_2)-\frac{(\ell-1)\nu_1-\sum_{j=2}^{\ell}\nu_j}{C(n-2,\ell-1)+\ell} = \frac{1}{n}(\nu_1-\nu_2)-\frac{1}{C(n-2,\ell-1)+\ell}\Big(\sum_{j=2}^{\ell}(\nu_1-\nu_j)\Big)$$
$$\ge \frac{1}{n}(\nu_1-\nu_2)-\frac{1}{C(n-2,\ell-1)+\ell}\Big(\sum_{j=2}^{\ell}(j-1)(\nu_1-\nu_2)\Big)$$
$$= \frac{1}{n}(\nu_1-\nu_2)-\frac{1}{C(n-2,\ell-1)+\ell}\Big(\frac{\ell(\ell-1)}{2}(\nu_1-\nu_2)\Big),$$

where we used Inequality (12) for the deriving the first inequality.[16]

Since $\nu_1 > \nu_2$, the above difference in expected welfare is non-negative if

$$\frac{1}{n} \geq \frac{\ell}{2} \frac{\ell - 1}{C(n-2, \ell-1) + \ell}. \tag{13}$$

This is equivalent to showing

$$\frac{C(n-2, \ell-1) + \ell}{(\ell-1)} \geq \frac{n\ell}{2}.$$

For $n = 10$ and $n = 11$, we have $\ell = 4$, and the above inequality is satisfied – note that it satisfied strictly for $n = 11$. So, assume $n \geq 12$. Now, by Fact 1 in the supplementary appendix and Proposition 2, we know that for $n \geq 12$,

$$\frac{C(n-2, \ell-1) + \ell}{(\ell-1)} \geq \frac{C(n-2, 3) + 4}{3} = \frac{(n-2)(n-3)(n-4) + 24}{18},$$

and

$$\frac{n+1}{2} \geq \ell.$$

As a result, it is sufficient to show that

$$\frac{(n-2)(n-3)(n-4) + 24}{18} \geq \frac{n(n+1)}{4}.$$

But this is true since

$$\begin{aligned} 2(n-2)(n-3)(n-4) + 48 - 9n(n+1) &= 2n^3 - 27n^2 + 43n \\ &= n(2n^2 - 27n + 43) \\ &= n\big((2n-3)(n-12) + 7\big) \\ &> 0, \end{aligned}$$

where for the last inequality, we used the fact that $n \geq 12$. Notice that the inequality is strict. Hence, for $n \geq 10$, the expected welfare from the r-optimal mechanism is higher than the GL mechanism with strict inequality holding for $n \geq 11$. □

A.7. Proofs of Theorem 4 and Theorem 5

In this section, we give proofs of Theorem 4 and Theorem 5. We first show that every r-Pareto optimal allocation rule satisfies the fact that probabilities add up to 1, i.e., the good is never wasted.

Lemma 6. *If f is an r-Pareto optimal or an r-optimal ranking allocation rule with probabilities $(\pi_1, \ldots, \pi_n)$, then*

$$\sum_{i \in N} \pi_i = 1.$$

Proof. Suppose f is a ranking allocation rule with probabilities $(\pi_1, \ldots, \pi_n)$. Assume for contradiction f is r-optimal but $\sum_{i=1}^n \pi_i < 1$. Let $\delta = 1 - \sum_{i \in N} \pi_i > 0$. We construct another ranking allocation rule f' with probabilities $\pi'_i \equiv \pi_i + \frac{\delta}{n}$ for all $i \in N$. Note that $\sum_{i \in N} \pi'_i = 1$ and

$$\begin{aligned} \sum_{k=1}^n (-1)^k C(n-1, k-1)\pi'_k &= \sum_{k=1}^n (-1)^k C(n-1, k-1)\pi_k + \frac{\delta}{n} \sum_{k=1}^n (-1)^k C(n-1, k-1) \\ &= \sum_{k=1}^n (-1)^k C(n-1, k-1)\pi_k \\ &= 0, \end{aligned}$$

where the first equality is from the definition of $(\pi'_1, \ldots, \pi'_n)$, the second equality follows from the fact that $\sum_{k=1}^n (-1)^k \times C(n-1, k-1) = 0$, and the third equality follows from Proposition 1 and the fact that $(\pi_1, \ldots, \pi_n)$ is a satisfactorily

[16] Note that the above inequalities are equalities for the uniform distribution since Inequality (12) holds with equality for the uniform distribution.

implementable ranking allocation rule. Hence, by Proposition 1, f' is satisfactorily implementable. But this contradicts the r-optimality of f.

Now, suppose f is r-Pareto optimal. The above argument also implies that at every valuation profile $\mathbf{v}$, we have

$$\sum_{i \in N} v_i f'_i(\mathbf{v}) \geq \sum_{i \in N} v_i f_i(\mathbf{v}),$$

with strict inequality holding at almost everywhere. This contradicts the fact that f is r-Pareto optimal. $\square$

This leads to a simplification of r-Pareto optimality in terms of first-order stochastic-dominance.

Definition 10. A ranking allocation rule f with probabilities $(\pi_1, \ldots, \pi_n)$ **first-order stochastic-dominates (FOSD)** a ranking allocation rule f' with probabilities $(\pi'_1, \ldots, \pi'_n)$ if for every $j \in N$, we have

$$\sum_{i \leq j} \pi_i \geq \sum_{i \leq j} \pi'_i,$$

with strict inequality holding at least once. In this case, we write $f \succ_{FOSD} f'$.

Lemma 7. *Suppose f is a ranking allocation rule with probabilities $(\pi_1, \ldots, \pi_n)$ such that it is satisfactorily implementable. Then, f is r-Pareto optimal if and only if*

1. $\sum_{i \in N} \pi_i = 1$ *and*
2. *if there exists no ranking allocation rule f' with probabilities $(\pi'_1, \ldots, \pi'_n)$ such that*
   - $\sum_{i \in N} \pi'_i = 1$,
   - *f' is satisfactorily implementable, and*
   - $f' \succ_{FOSD} f$.

Proof. Suppose a ranking rule f with probabilities $(\pi_1, \ldots, \pi_n)$ is r-Pareto optimal. By Lemma 6, we know that $\sum_{i \in N} \pi_i = 1$. Now, assume for contradiction that there exists a ranking allocation rule f' with probabilities $(\pi'_1, \ldots, \pi'_n)$ such that f' is satisfactorily implementable, $\sum_{i \in N} \pi'_i = 1$, and $f' \succ_{FOSD} f$. Since $f' \succ_{FOSD} f$, for any profile of generic valuations $\mathbf{v}$ with $v_1 > v_2 > \ldots > v_n$, we have

$$\sum_{i \in N} v_i \pi'_i \geq \sum_{i \in N} v_i \pi_i.$$

The strict inequality must hold for some generic valuation profile by the definition of first-order stochastic dominance. Now, take any arbitrary valuation profile $\mathbf{v}$. Note that the total welfare of a ranking allocation rule is continuous in the valuations of the agents. Hence, it can be written as a limit point of generic valuation profiles like above. This implies that for every valuation profile $\mathbf{v}$, we have

$$\sum_{i \in N} v_i f'_i(\mathbf{v}) \geq \sum_{i \in N} v_i f_i(\mathbf{v}),$$

with strict inequality holding for some $\mathbf{v}$. This implies that f is not r-Pareto optimal, a contradiction.

Now, for the other direction suppose f is a ranking allocation with probabilities $(\pi_1, \ldots, \pi_n)$ satisfying the properties in the claim. Assume for contradiction that f is not Pareto optimal. Then, there exists a satisfactorily implementable ranking allocation rule f' with probabilities $(\pi'_1, \ldots, \pi'_n)$ such that for valuation profiles $\mathbf{v}$, we have

$$\sum_{i \in N} v_i f'_i(\mathbf{v}) \geq \sum_{i \in N} v_i f_i(\mathbf{v}),$$

with strict inequality holding for some $\mathbf{v}$. By Lemma 6, we can assume $\sum_{i \in N} \pi'_i = 1$ without loss of generality. For generic valuation profiles $\mathbf{v}$ with $v_1 > \ldots > v_n$, we have $\sum_{i \in N} v_i \pi'_i \geq \sum_{i \in N} v_i \pi_i$. As in the previous paragraph, continuity of the total welfare of agents in a ranking allocation rule implies that $f' \succ_{FOSD} f$. This is a contradiction. $\square$

We now provide a proof of Theorem 4.

Proof of Theorem 4. We denote the GL allocation rule as f^G. Assume for contradiction that f^G is not r-Pareto optimal. By Lemma 7, there is another ranking allocation rule f such that f is satisfactorily implementable and $f \succ_{FOSD} f^G$. Suppose the allocation probabilities of f are $(\pi_1, \ldots, \pi_n)$. We know that the allocation probabilities of f^G are $(1 - 1/n, 1/n, 0, 0, \ldots, 0)$. Since $f \succ_{FOSD} f^G$, $\pi_1 + \pi_2 = 1$, and hence, $\pi_3 = \ldots = \pi_n = 0$. Since f is satisfactorily implementable, by Proposition 1, we get

$$\pi_1 - (n-1)\pi_2 = 0.$$

Using $\pi_1 + \pi_2 = 1$ and simplifying, we get $\pi_1 = 1 - 1/n$. Hence, f is the Green–Laffont allocation rule, which is a contradiction.

The above proof along with Lemma 6 also makes it clear that among all ranking allocation rules which allocates probability to only π_1 and π_2, the GL allocation rule is the unique r-Pareto optimal allocation rule. □

We now provide a proof of Theorem 5.

Proof of Theorem 5. Suppose $n \leq 8$. Then, the GL allocation rule is an r-optimal allocation rule by Proposition 2. Since $\pi_1 + \pi_2 = 1$ in the GL allocation rule, this implies that the GL allocation rule dominates every other satisfactorily implementable ranking allocation rule an FOSD sense. By Lemma 7, the GL allocation rule is the unique r-Pareto optimal allocation rule.

Suppose $n > 8$. Then, Theorem 1 implies that there is a unique r-optimal allocation rule. Hence, no other satisfactorily implementable ranking allocation rule can dominate this unique r-optimal allocation rule in an FOSD sense. By Lemma 7, this unique r-optimal allocation rule is then r-Pareto optimal.

Finally choose an r-Pareto optimal allocation rule $(\pi_1, \ldots, \pi_n)$. By definition of π_1^*, we have $\pi_1 \leq \pi_1^*$. Further, if $\pi_1 < 1 - 1/n$, the GL allocation rule dominates this allocation rule in an FOSD sense, and by Lemma 7, it is not r-Pareto optimal. Hence, $\pi_1 \geq 1 - 1/n$. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.geb.2017.07.002.

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