Welfare and Incentives in Partitioned School Choice Markets*

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Abstract

Divided enrollment systems cause school assignments to be *unfair* and *wasteful*. Iterative version of the student-optimal stable mechanism (I-SOSM), proposed by *Manjunath and Turhan* (2016), achieves *individually rational* and *fair* assignments in such partitioned school choice markets for any number of iteration. It also reaches a *non-wasteful* matching when iterated sufficiently many times. This paper examines the effects of *partition structure* of schools on students' welfare and incentives they face under I-SOSM. I find that when school partition gets coarser students' welfare weakly increases under I-SOSM for any number of iteration. I also show that under coarser school partitions I-SOSM becomes weakly less manipulable for students according to a criteria defined by *Pathak and Sönmez (2013)*. These results suggest that when full integration is not possible keeping school partition as coarse as possible benefits students with respect to their welfare and incentives they face if stability is a concern for policymakers.

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1 Introduction

In almost every city schools are broken into groups that compete for the same crop of school-aged children. Public schools, charter schools, private schools and voucher programs are main school groups in USA which are antagonistic towards one another. Most families participate in admissions processes in multiple school groups. In one of their recent reports[1] the *Center on Reinventing Public Education (CRPE)* points the issue of parallel assignments as follows:

> "*Families in many districts can choose from a variety of charter and district schools for their children. But to make these choices, parents often must fill out multiple application forms and, navigate schools that may have different requirements, deadlines, and selection preferences such as sibling attendance or proximity to the school. Once parents complete the applications and schools make offers, some families receive multiple offers and often hold on to them until the last minute, while other families receive a few or no offers, remaining on waitlists well into the fall. Not only this is difficult for families, it favors families with the time and knowledge to navigate its inherent complexities.*"

In New Orleans, for instance, there are two different large bodies: the Orleans Parish School Board (PSB) and the Recovery School District (RSD). There have been attempts to unify these two institution in order to have common enrollment process known as *"OneApp"*. The *OneApp* process began by requiring only the district-run and charter schools overrun by the RSD. Schools operated by the PSB including selective admissions schools, local charter schools and schools accepting vouchers for eligible students participate in the *OneApp* "*voluntarily*". Now, *OneApp* accommodates 89% of the schools but not all of them.

In many other cities full integration of school systems is not possible, as well. The process of matching students to different groups of schools are independent of one another. It is easy to see that this leads to inefficiencies: seats in one school system are "*wasted*" on students who choose to go to one of the other systems. Consequently, at the end, many children might be left unmatched or matched to schools that they find worse than others with empty seats. *Dreilinger (2013)* reports that many students simply do not show up on the first day of classes in New Orleans because they receive multiple acceptances from different schools. In Milwaukee, many students are left to participate in what is called "*open enrollment*" after the main round of admissions because of the same reason. This is an uncoordinated and chaotic process where parents apply to individual schools in search of a seat for their child.

[1] See "Coordinating Enrollment Across School Sectors: An Overview of Common Enrollment Systems" (2014) by the Center on Reinventing Public Education.

This way of matching students to schools is inefficient and leads to violations of priorities that students have at schools.

When full integration is not possible, *Manjunath and Turhan* (2016) propose a way for groups of schools to match and re-match. Each school group uses the student-optimal stable mechanism (SOSM). Once various groups of schools separately match students using the SOSM, the problem of wasted seats persists. The iterative process described to solve this problem is as follows: Each student submits to each group of schools her preferences over those schools. Each group separately apply the SOSM restricted to that group of schools. A student who is matched to a school in more than one group accepts his most preferred among them and turns down the rest. This is the initial match. There may be wasted seats at this point since some of the matches are turned down. Each group of school computes a re-match in the following way:

1. If a student accepted a seat offered by this group, his preferences over this group are *"truncated" at* the school which was offered to him. If a student turns down the seat offered by this group, his preferences over this group are truncated *above* the school that was offered to him.

2. The group *"re-matches"* using the SOSM restricted to this group for the truncated preferences.

Again, every student who is matched to a school by more than one group accepts his most preferred among them and turns down the rest. This is the "re-match". The re-match may still have wasted seats so the re-matching process may be repeated. If it is repeated enough times, the resulting matching will be stable, i.e., individually rational, non-wasteful, and eliminates justified envy.[2] While iterating may be costly, as it involves asking inputs from students, *Manjunath and Turhan (2016)* show, both analytically and via simulations, that there are significant gains from the first few iterations.

This paper studies effects of schools' partition structure on students' welfare and incentives they face under the rule described above. I show that for any given preference profile the above procedure yields a (weakly) Pareto superior outcome under a coarser school partition for any number of iteration *(Theorem 3).* When each school group runs the SOSM only once the procedure is strategy proof *(Theorem 2)* so the Pareto comparison *(Theorem 1)* is robust. If each school iterates the SOSM more than once strategy-proofness is lost *(Example 1).* When iterated sufficiently many times the *I-SOSM* rule gives a stable matching. I compare the manipulability of two such stable *I-SOSM* rules, one is under finer partition than

[2]See Balinski & Sönmez (1999) for the decomposition of stability into individual rationality, fairness and non-wastefulness.

another, and find that *I-SOSM* under a finer school partition is *weakly more manipulable for every student* than the one under a coarser school partition *(Theorem 4)*. Lastly, I find that *dropping strategies* are exhaustive under *I-SOSM,* for any number of iteration, in the sense that for any manipulation strategy there exists a dropping strategy which achieves it *(Theorem 5).*

1.1 Related Literature

This paper is related with *Ekmekci and Yenmez (2015).* They provide a new framework to study the incentives of a school to join a clearinghouse and show that each school prefers to remain out of the centralized system when others join and either version of the deferred acceptance algorithm is used by the central clearinghouse. Their analysis provides an explanation of why some charter schools have evaded the clearinghouse. They propose two schemes that can be used by policymakers in order to overcome this issue. Theirs is an equilibrium analysis where each school can choose either to participate in or to evade from the centralized clearinghouse.

Another related work is *Doğan and Yenmez (2017).* They study divided and unified enrollment systems for student assignment when there are different types of schools. They show that the unified enrollment system is better for students than the divided enrollment system in the context of student assignment in Chicago to selective and nonselective schools. It is important to note that the mechanism they study in the 2-stage divided system is different than the one I consider.

This work is also closely related to *Pathak and Sönmez (2013) and Chen, Egesdal, Pycia and Yenmez (2016).* I use manipulability comparison criterion, which is introduced in *Pathak and Sönmez (2013),* in order to compare manipulability of two *I-SOSMs* where one is under coarser school partition than the other one. *Chen, Egesdal, Pycia and Yenmez (2016)* study the manipulability of stable matching mechanisms and show that manipulability comparisons are equivalent to preference comparisons. They find that for any given agent a mechanism is more manipulable than another if and only if this agent prefers the latter to the former.

The remainder of the paper is organized as follow. In section 2, I formally define the model of school choice problem and the *I-SOSM* for a given partition of schools. In the section 3, I present results. Section 4 concludes. All proofs are in Appendix.

2 The Model

A **school choice problem in partitioned markets** consist of the following elements:

- a finite set of students $I = \{i_1, ..., i_n\}$,
- a finite set of schools $S = \{s_1, ..., s_m\}$ that is partitioned into K subsets $\{S^k\}_{k \in K}$,
- a strict priority structure of schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is a linear order over $I \cup \{\emptyset\}$ where $\emptyset \succ_s i$ means student i is *unacceptable* to school s,
- a capacity vector $q = (q_{s_1}, ..., q_{s_m})$ where q_s is the number of available seats at school s,
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's preference relation over $S \cup \{\emptyset\}$ where $\emptyset$ denotes the option of being unassigned.

The *at-least-as-well-as* relation R_i is obtained from P_i as follows: sR_is' if and only if either sP_is' or $s = s'$.

I denote the set of all preference relations over $S \cup \{\emptyset\}$ by $\mathcal{P}$. Given $P_i \in \mathcal{P}$, I denote its restriction to $S^k \cup \{\emptyset\}$ by P_i^k. For each $k \in K$, $\mathcal{P}^k$ is the set of preference relations over $S^k \cup \{\emptyset\}$.

For each $i \in I$, $U(P_i, s)$ is the weak upper contour set of P_i at $s \in S \cup \{\emptyset\}$ and $\overline{U}(P_i, s)$ is the strict upper contour set of P_i at $s \in S \cup \{\emptyset\}$. That is,

$$U(P_i, s) = \{s' \mid s'R_is\}, \ and \ \overline{U}(P_i, s) = \{s' \mid s'P_is\}.$$

For each $k \in K$ and each $s \in S^k \cup \{\emptyset\}$, I define $U(P_i^k, s)$ and $\overline{U}(P_i^k, s)$ similarly.

The admission policy of each school $s \in S$ is represented by a **choice rule** $C^s : 2^I \times \{q_s\} \rightarrow 2^I$ which maps each nonempty set $J \subseteq I$ of students to a subset $C^s(J; q_s) \subseteq J$ such that $\mid C^s(J; q_s) \mid \leq q_s$. I assume that for each school $s \in S$, $C^s(\cdot; q_s)$ is **responsive** to the priority ranking $\succ_s$, i.e., for each $J \subseteq I$, $C^s(J; q_s)$ is obtained by choosing the highest-priority acceptable students in J until q_s students are chosen or no acceptable student is left.

A **matching** $\mu : I \longrightarrow S \cup \{\emptyset\}$ is a function such that for each $s \in S$, $\mid \mu^{-1}(s) \mid \leq q_s$. From now on, I will omit the superscript "-1" and use $\mu(s)$ to mean $\mu^{-1}(s)$. The set of all matchings are $\mathcal{M}$.

A matching μ **Pareto dominates** matching ν if $\mu(i)R_i\nu(i)$ for all $i \in I$ and $\mu(i)P_i\upsilon(i)$ for some $i \in I$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

A matching μ is **individually rational** at $P \in \mathcal{P}$ if for each $i \in I$, $\mu(i)R_i\emptyset$. That is, no student is assigned to a school that she finds worse than being unassigned. A matching μ is $\mathbf{non-wasteful}$ at $P \in \mathcal{P}$ if there does not exist a student $i \in I$ and a school $s \in S$ such that $sP_i\mu(i)$ and $\mid \mu(s) \mid < q_s$. That is, a seat is wasted if it is left empty despite there being a student who prefers it to what she is matched to. A matching μ is **fair** at $P \in \mathcal{P}$ if for all $i \in I$ and $s \in S$ such that $sP_i\mu(i)$, for each $j \in \mu(s)$, $j \succ_s i$. That is, priorities are respected. A matching is **stable** if it is individually rational, non-wasteful and fair.

A **mechanism** φ is a function that maps preference profiles to matchings. The matching under φ at preference profile P is denoted by $\varphi(P)$ and student i's match is denoted by $\varphi_i(P)$ for each $i \in I$.

A mechanism φ is **individually rational** (**non – wasteful**) [**fair**] if $\varphi(P)$ is an individually rational (non-wasteful) [fair] matching for all $P \in \mathcal{P}^{|I|}$.

A mechanism φ is said to be **strategy proof** if there does not exist a preference profile P, an agent $i \in I$, and preference $P_i^{'}$ of agent i such that

$$\varphi_i(P_i^{'}, P_{-i})P_i\varphi_i(P).$$

That is, no student has an incentive to misreport her preferences under the mechanism.

As this paper analyzes the effect of partition structure of schools on welfare of students as well as the incentives they face under the iterative version of the student-optimal stable mechanism (SOSM), it is useful to start with considering the case with no partitioning of schools first. Gale and Shapley (1962) shows that there exists a stable (i.e., individually rational, non-wasteful and fair) matching for every problem by using the following **student – proposing deferred acceptance algorithm** (SPDA):

- *Step 1:* Each student applies to her top choice school. Each school rejects the lowest-ranking students in excess of its capacity and all unacceptable students among those who applied to it, keeping the rest of the students temporarily (so students not rejected at this step may be rejected in later steps).

- *Step k (k$\geq$ 2):* Each student who was rejected in Step (k-1) applies to her next highest choice (if any). Each school considers these students *and* students who are temporarily held from the previous step together, and rejects the lowest-ranking students in excess of its capacity and all unacceptable students, keeping the rest of the students temporarily.

The algorithm terminates at a step in which no rejection occurs. The algorithm always terminates in a finite number of steps. Gale and Shapley (1962) show that the resulting matching is stable. The SOSM is strategy proof for students (Dubins and Freedman, 1981; Roth, 1982). The outcome of the SPDA is the *student-optimal stable matching* that every student likes at least as well as any other stable matching.

2.1 The Iterative Student Optimal Stable Mechanism (I-SOSM)

For each $k \in K$, the procedure the k^{th} group of schools, S^k, uses to match students to schools in S^k takes as an input students' preferences over S^k and schools' priorities and returns a match in $\mathcal{M}^k$. This defines an $S^k - rule$.

When each group of schools is matched to students by using its own rule, each student may be matched to more than one school. Then, let each such student to be matched her favorite among them. For a set of matchings $\mathcal{M}' \subset \mathcal{M}$ and a profile of student preferences $P \in \mathcal{P}$, the "*combination*" of $\mathcal{M}'$ at P, which is denote by $\mathcal{C}(\mathcal{M}', P)$, is $f : I \to S \cup \{\emptyset\}$ such that for each $i \in I$, $f(i)$ is the most favorite school of i among those she is matched to by matches in $\mathcal{M}'$.

For each $k \in K$, let ψ^{k-SO} be the student-optimal stable $S^k - rule$. Define φ^{T-SO} by setting for each $P \in \mathcal{P}$, $\varphi^{T-SO}(P) = \mu_T$ where μ_T is computed as follows:

- For each $i \in I$, $\mu_0(i) = \emptyset$,
- For each $t \in \{1, ..., T\}$, $(S_i^t)^k = \begin{cases} U(P_i^k, \mu_{t-1}^k(i)) & if\ \mu_{t-1}(i) = \mu_{t-1}^k(i) \\ \overline{U}(P_i^k, \mu_{t-1}^k(i)) & if\ \mu_{t-1}(i) \neq \mu_{t-1}^k(i) \end{cases}$ and $(P_i^t)^k = P_i\ |_{(S_i^t)^k}$,
- For each $k \in K$, $\mu_t^k \equiv \psi^{k-SO}((P^t)^k)$, and $\mu_t \equiv \mathcal{C}(\{\mu_t^k\}_{k \in K}, P)$.

An interpretation of this computation is as follows: For each $i \in I$, for each $k \in K$, her preference P_i^k is generated from her original preference P_i. Then, for each group of schools, the SPDA outcome is computed. If a student receives more than one school assignment from the matches in different group of schools, she is given the most preferred one according to her original preference P_i. This is the initial match. Each group of schools computes the re-match in the following way:

1. If a student is held by a school in this group, her preferences over this group are "truncated" *at* the school which was offered to her. That is, every other school that she finds worse than what she has been offered is considered unacceptable. If a student turned down the school offered by this group, her preferences over this group are truncated *above* the school that was offered to her. That is, the school that was offered to her and every school worse than it is deemed as unacceptable.

2. The group "re-matches" using the SOSM restricted to that group for the truncated student preferences.

3. Groups re-match until no student is matched to more than one school.

This process yields an individually rational, non-wasteful and fair match if it is iterated sufficiently many times. If it is iterated a few times only, the resulting matching is individually rational and fair, but might be wasteful. The nice property of this process is that increasing the number of iterations leads to Pareto improvements. In fact, fewer students are unmatched as the number of iterations increase.

3 Results

I fix I, S, $q = (q_s)_{s \in S}$, and $\succ = (\succ_s)_{s \in S}$ and let school partition structure to vary. A problem is denoted by a pair (α, P) where α is a school partition and P is a preference profile of students. Let $\mathcal{Q}$ be the set of all partitions of S. Let $F^l : \mathcal{Q} \times \mathcal{P}^{|\mathcal{I}|} \longrightarrow \mathcal{M}$ be a mapping where the outcome $F^l(\alpha; P)$ is computed by iterating the I-SOSM l times under school partition α at student preference profile P.

Definition 1. A partition $\alpha = \{S^1, ..., S^K\}$ of the set S is **finer than** partition $\beta = \{\tilde{S}^1, ..., \tilde{S}^L\}$ of the same set S, if every element of the partition α is a subset of some element of partition β, i.e., for all $i \in \{1, ...K\}$ there exist some $j \in \{1, ..., L\}$ such that $S^i \subseteq \tilde{S}^j$.

This *"finer-than"* relation on the set of partitions of S is a *partial order.* Each set of elements has a least upper bound and a greatest lower bound, so that it forms a lattice. For a set of schools $S = \{s_1, ..., s_m\}$, the partition $\{\{s_1\}, \{s_2\}, ..., \{s_m\}\}$ is the *finest* partition of S and the partition $\{\{s_1, ..., s_m\}\}$ is the *coarsest* partition of S.

Fact 1. *With the finest partition, given a profile of student preferences P, the iterative student proposing deferred acceptance outcome is equivalent to the outcome of the school proposing deferred acceptance outcome under the coarsest partition at P.*

I-SOSM achieves the extremal matchings under the finest and coarsest school partitions, respectively. Under the coarsest partition it finds the student optimal stable matching when iterated only once. However, under the finest partition it finds the school optimal stable matching of the unified market when iterated sufficiently many times. For partitions other than finest and coarsest ones I-SOSM may find stable matchings which are not extremal ones when iterated sufficiently many times.

3.1 Welfare comparison

For a given student preference profile P, let L be the maximum number of iterations needed for the iterative student-proposing deferred acceptance algorithm to find individually rational, non-wasteful and fair matching (i.e., stable matching) at P. *Manjunath and Turhan (2016)* prove the existence of such a finite integer. If $l = 1$, i.e., school groups match students to their respective schools by using the student-optimal stable mechanism but do not re-match again after students pick which offer to accept, then the resulting matching under a school partition Pareto dominates the resulting matching under a finer school partition.[3]

Theorem 1. *Given two school partitions α and β such that α is finer than β, $F^1(\beta; P) R_S F^1(\alpha; P)$ for every problem $P \in \mathcal{P}$.*

Next, I show that students cannot gain by misreporting under this rule when $l = 1$. Hence, the Pareto comparison in Theorem 1 is robust.

Theorem 2. *For any given partition $\alpha \in \mathcal{Q}$, $F^1(\cdot; \alpha)$ is strategy proof.*

For a given preference profile $P \in \mathcal{P}^{|\mathcal{I}|}$, if we consider two different school partitions where one is finer than another one and iterate the iterative student-proposing deferred acceptance algorithm same number of rounds under both partitions, then the outcome of the iterative student-proposing deferred acceptance algorithm under the coarser partition Pareto dominates the outcome of the iterative student-proposing deferred acceptance algorithm under the finer partition. The Pareto comparison result for $l = 1$ can be generalized for any arbitrary number of iteration. I formally state this as follows:

Theorem 3. *Given two school partitions α and β such that α is finer than β, $F^l(P; \beta) R_S F^l(P; \alpha)$ for every problem $P \in \mathcal{P}$ and every $l \geq 1$.*

The direct corollary of Theorem 3 is that if iterated sufficiently many times under school partitions α and β, where the former one is finer than the latter, outcomes of two *stable* rules can be compared in the Pareto sense. Let $F(P; \alpha)$ and $F(P; \beta)$ be two individually rational, fair and non-wasteful (i.e., stable) matchings at $P \in \mathcal{P}^{|I|}$ computed through iterative student proposing deferred acceptance under school partitions α and β, respectively.

Corollary 1. *Given two school partitions α and β such that α is finer than β, $F(P; \beta) R_S F(P; \alpha)$ for every problem $P \in \mathcal{P}$.*

Corollary 1 has the following interesting interpretation: School partitions form a lattice with respect to "finer than" relation. The set of stable matchings forms a lattice with respect to common student preferences. One can define a mapping from the set of all possible school partitions in to the set of all stable matchings in a way that given a school partition I-SOSM

[3]Our Theorem 1 implies Theorem 1 of Doğan and Yenmez (2017). Their Theorem 1 is a Pareto comparison result between the coarsest school partition (they call it "unified enrollment system") and a given finer school partition.

(iterated sufficiently many times) maps it to a stable matching. Corollary 1 states that this mapping is *weakly monotonic* in the following sense. When two partitions, one is finer than the other one, are considered the stable matching I-SOSM finds under the coarser one is weakly preferred by every student than the one I-SOSM finds under the finer one.

When school groups iterate the SOSM more than once Pareto comparison result in Theorem 3 need not to be robust because, then, strategy proofness is lost. To see it consider the following example:

Example 1. Let $I = \{i, j\}$ and $S = \{a, b\}$ where $S^1 = \{a\}$ and $S^2 = \{b\}$ with $q_a = q_b = 1$. Student preferences and school priorities are as follows:

$$\begin{array}{ll} P_i: \ a - b - \emptyset & \succ_a: \ j - i \\ P_j: \ b - a - \emptyset & \succ_b: \ i - j \end{array}$$

The outcome of the iterative student-proposing deferred acceptance algorithm is $\begin{pmatrix} i & j \\ b & a \end{pmatrix}$ since both students apply to both schools and schools pick their favorite students. Now consider the following preference profile where student j misreport his preference:

$$\begin{array}{ll} P_i: \ a - b - \emptyset & \succ_a: \ j - i \\ \tilde{P}_j: \ b - \emptyset & \succ_b: \ i - j \end{array}$$

In the first iteration both students apply to school b and only student i applies to school a. Student i gets both school a and school b. But, he chooses school a and declines school b. Then, his preference is truncated at S^2 in a way that he does not find any school in S^2 acceptable. In the second iteration, the only student who apply to school b is student j and he receives it. Then, the outcome of student proposing deferred acceptance iterated twice is $\begin{pmatrix} i & j \\ a & b \end{pmatrix}$. Hence, student j is better off by misreporting.

3.2 Incentive comparison

In this section, I fix I, S, $q = (q_s)_{s \in S}$, and $\succ = (\succ_s)_{s \in S}$. Therefore, a pair (α, P) defines a problem. As seen in Example 1 given a school partition α, an *I-SOSM* mechanism $F^l(\cdot; \alpha)$ need not be strategy proof. In this section, I analyze the effect of partition structure of schools on incentives that students face under *I-SOSM*.

Definition 2. A mechanism φ is **manipulable** by student i at problem P if there exists a preference $P_i^{'} \in \mathcal{P}_i$ for this student such that $\varphi_i(P_i^{'}, P_{-i}) P_i \varphi_i(P)$.

To compare two manipulable mechanisms we borrow the following definitions from *Pathak and Sönmez (2013).* This is an *agent-by-agent* manipulability comparison of two manipulable mechanisms.

Definition 3. A mechanism ψ is **as strongly manipulable as** mechanism ϕ if for any profile P at which mechanism ϕ is manipulable ψ is manipulable by any player who can manipulate φ.

Manipulability comparisons of two stable *I-SOSMs*, one is under finer school partition than another, are equivalent to partition comparisons: whenever a student is able to manipulate a stable *I-SOSM* under a certain school partition, the same student is able to manipulate it under a finer school partition.

Theorem 4. *Let $F(\cdot;\alpha)$ and $F(\cdot;\beta)$ be iterative-SOSMs under partitions α and β, respectively, where α is finer than β. Then, $F(\alpha;\cdot)$ is as strongly manipulable as $F(\beta;\cdot)$.*

3.3 How to manipulate I-SOSM

To study manipulations under *I-SOSM*, one way to begin may be to consider all possible strategies of a particular student. Fortunately, there is a type of manipulation that is exhaustive for *I-SOSM* mechanism.

Definition 4. (*Kojima and Pathak, 2009)* A *dropping* strategy of a student $i \in I$ is an ordered list$P_i^{'}$ obtained from P_i by removing some acceptable schools, i.e., not necessarily a tail of least preferred schools. Formally, for a student i with preferences P_i over schools and remaining unmatched, $P_i^{'}$ is a dropping strategy if for any schools $s, s^{'} \in S$, $sR_i^{'}s^{'}R_i^{'}\emptyset$ implies $sR_is^{'}R_i\emptyset$.

The last result states that dropping strategies are exhaustive under *I-SOSM.*

Theorem 5. *Given a school partition α consider an I-SOSM $F^l(\cdot;\alpha)$ for some $l \geq 2$. Fix preferences of students other than student i. Suppose the mechanism produces μ under some report of student i. Then, there exists a dropping strategy that produces a matching that student i weakly prefers to μ according to her true preferences.*

Kojima and Pathak (2009) present a similar results for *stable mechanisms.* Notice that I-SOSM $F^l(\cdot;\alpha)$ for some $l \geq 2$ need not be stable. It might be wasteful. Hence, Theorem 5 is not implied by their result.

4 Conclusion

Building on *Manjunath and Turhan (2016),* this papers studies the effects of partition structures of schools under the I-SOSM mechanism which finds an individually rational and fair assignment for any number of iteration and also non-wasteful assignment if iterated sufficiently many times. I find that, for a given problem, as the schools partition gets finer, under the *I-SOSM* students get weakly worse off. Also, as the school partition gets finer I-SOSM becomes at least as manipulable when iterated sufficiently many times to find a stable outcome. These results suggest that policy makers should prevent schools to form finer partitions if stability is desirable for them. I also find that dropping strategies for students are exhaustive under I-SOSM for any number of iteration.

5 Appendix

Proof. (Theorem 1) Consider two school partitions $\gamma^1 =\{\tilde{S}^1, \bar{S}^1, S^2..., S^K\}$ and $\gamma^2 = \{S^1, S^2, ..., S^K\}$ where $\tilde{S}^1 \subset S^1$ and $\bar{S}^1 \subset S^1$ and $\tilde{S}^1 \cup \bar{S}^1 = S^1$ so that partitions γ^1 and γ^2 have $K-1$ same sets S^2,...,S^K but set of schools S^1 in partition γ^2 is divided into two disjoint sets $\tilde{S}^1$ and $\bar{S}^1$ in partition γ^1. Given a preference profile P, students submit the same preference relations to school groups S^2, S^3,...,S^K, namely, P^2, P^3,..., P^K. Then, ψ^{k-SO} yields the same matchings under partitions γ^1 and γ^2, i.e., $\psi^{k-SO}(\gamma^1; P) = \psi^{k-SO}(\gamma^2; P)$ for all $k = 2, 3, ..., K$. Students submit the preference profile P^1 to the school group S^1 under partition structure γ^2. Since both $\tilde{S}^1$ and $\bar{S}^1$ are subsets of S^1, the preference relations, for each student, submitted to groups $\tilde{S}^1$ and $\bar{S}^1$, $P^1 \mid_{\tilde{S}^1}$ (P^1 restricted to $\tilde{S}^1$) and $P^1 \mid_{\bar{S}^1}$ (P^1 restricted to $\bar{S}^1$) agree with the preference relation submitted to group S^1, P^1. Then, by the Theorem 2 of *Gale and Sotomayor (1985),* we have both $\psi^{1-SO}(P^1)R_i\psi^{1-SO}(P^1 \mid_{\tilde{S}^1})$ and $\psi^{1-SO}(P^1)R_i\psi^{1-SO}(P^1 \mid_{\bar{S}^1})$. Then, for each student $i \in I$, we have:

$$\mathcal{C}(\{\psi^{1-SO}(P^1), ..., \psi^{K-SO}(P^K)\}; P)R_i\mathcal{C}(\{\psi^{1-SO}(P^1 \mid_{\tilde{S}^1}), \psi^{1-SO}(P^1 \mid_{\bar{S}^1}), \psi^{2-SO}(P^2), ..., \psi^{K-SO}(P^K)\}; P)$$

Given two partitions α and β such that α is finer than β, we can construct a sequence of partitions γ^1, γ^2,...,γ^b such that $\alpha = \gamma^1$, $\beta = \gamma^b$ and two consecutive partitions γ^t and γ^{t+1} are such that only one set in γ^{t+1} is divided into two disjoint subsets in order to obtain the partition γ^t. Applying the procedure above gives us each student gets weakly better off under partition β than under partition α. $\square$

Proof. (Theorem 2): Let α be $\{S^1, ..., S^K\}$. The outcome is $\mathcal{C}(\{\psi^{1-SO}(P^1), ..., \psi^{K-SO}(P^K)\}; P)$ where each group of schools $k \in K$ uses student-optimal stable rule, ψ^{k-SO}, for a given preference profile P. Since each student knows that there will be no more iteration and ψ^{k-SO} is strategy proof it is weakly dominant strategy for each student to report P^k truthfully to each S^k. Then, for each student $i \in I$, given the schools (or no school option) he is offered by each group of schools $\psi_i^{1-SO}(P^1)$,...,$\psi_i^{K-SO}(P^K)$, student i will pick the best one of them according to her true preference. Hence, for each student $i \in I$, it is weakly dominant strategy to report P_i truthfully. $\square$

Towards the proof of Theorem 3 we first prove three lemmas. The first one nests Lemma 1 of *Doğan and Yenmez (2017) where the authors compare the coarsest partition with a given finer partition.*

Lemma 1. *Consider the case where $l = 1$. Let β be a coarser school partition than α. The set of students who has applied to a school at or before Step t of the student-proposing deferred acceptance algorithm (SPDAA) under partition β is a subset of those who have applied under partition α, for every t.*

Proof. Take a school $s \in S$. Let $s \in S^1$ under partition β and $s \in \tilde{S}^1$ under partition α, without loss of generality. Notice that $\tilde{S}^1 \subseteq S^1$. Let $I_\beta^t(s)$ and $I_\alpha^t(s)$ be the set of students who has applied to school s at or before Step t of the student-proposing deferred acceptance algorithm (DAA) under partition β and partition α, respectively. We want to show that $I_\beta^t(s) \subseteq I_\alpha^t(s)$. We prove this claim by mathematical induction on t.

When $t = 1$, in every partition, each student applies to his top overall ranked school, under both partition α and β. That is, $I_\beta^1(s) = \{i \mid sP_is^{'},\ \forall s^{'} \in S^1\ s.t.\ s \neq s^{'}\}$ and $I_\alpha^1(s) = \{i \mid sP_is^{'},\ \forall s^{'} \in \tilde{S}^1\ s.t.\ s \neq s^{'}\}$. Clearly, $I_\beta^1(s) \subseteq I_\alpha^1(s)$ because if s is a top choice for a student under partition β, then it must be a top choice at some school group under partition α for the same student.

Assume that the claim holds true for every number smaller than t. We now show it for t. Let j be a student who has applied to school s exactly at Step t under partition β. That is, $j \in I_\beta^t(s) \setminus I_\beta^{t-1}(s)$. We claim that $j \in I_\alpha^t(s)$. If school s is highest ranked among schools in $\tilde{S}^1$ with respect to $P_j\mid_{\tilde{S}^1}$ then $j \in I_\alpha^1(s)$ which implies $j \in I_\alpha^t(s)$. Otherwise, if school s is not the highest-ranked school in $\tilde{S}^1$, there exists a school $s^{'} \in \tilde{S}^1$ which is ranked right above s among schools in $\tilde{S}^1$ with respect to $P_j\mid_{\tilde{S}^1}$. Since $j \in I_\beta^t(s) \setminus I_\beta^{t-1}(s)$ school $s^{'}$ must have rejected student j at Step $t-1$ or before under partition β. That means $j \in I_\beta^{t-1}(s^{'})$ and $j \notin C_{s'}(I_\beta^{t-1}(s^{'}); q_{s'})$. By the inductive assumption we have $j \in I_\alpha^{t-1}(s^{'})$. Since $C_{s'}$ is responsive we have $j \notin C_{s'}(I_\alpha^{t-1}(s^{'}); q_{s'})$. Therefore, student j applies to school s at Step t or before under partition α. So, $j \in I_\alpha^t(s)$.

We show that every student in $I_\beta^t(s) \setminus I_\beta^{t-1}(s)$ is also in $I_\alpha^t(s)$. This together with the inductive assumption that $I_\beta^{t-1}(s) \subseteq I_\alpha^{t-1}(s)$ imply that $I_\beta^t(s) \subseteq I_\alpha^t(s)$. $\square$

Lemma 2. *Let β be a coarser school partition than α. Suppose student preferences are truncated as described in Section 2.1 in I-SOSM. Suppose $F^r(P;\beta)RF^r(P;\alpha)$ for $P \in \mathcal{P}$, for all $r = 1, ..., l$. Consider any student $i \in I$. If school $s \in S$ is still acceptable to student i with respect to his truncated preference under school partition β at the end of iteration l, then s must be acceptable to student $i \in I$ with respect to his truncated preferences under partition α at the end of iteration l.*

Proof. Consider student $i \in I$. Suppose that school $s \in S$ is still acceptable to student i with respect to his truncated preference under school partition β at the end if iteration t. That means student i obtains either school s or a school which is strictly worse than school s in S^k where $s \in S^k$ with respect to school partition β. From $\tilde{S}^r$, where $s \in \tilde{S}^r$ with respect to partition α, at the end of the deferred acceptance student i gets either the same school or a strictly worse one compared to what he gets in S^k under partition β with respect to his preference relation P_i. Hence if school $s \in S$ survives the truncation under partition β, it must survive truncation under partition α. Thus, every school which is still acceptable after the truncation under partition β must be acceptable after truncation under partition α. □

Lemma 3. *Let β be a coarser school partition than α. Consider iteration l of I-SOSM. Suppose that for every student any school which is still acceptable, i.e., survives truncation, at the end of iteration $(l-1)$ under partition β is also acceptable, i.e., survives truncation, at the end of iteration $(l-1)$ under partition α. Then, in iteration l, the set of students who has applied to a school at or before Step t of the SPDAA under partition β is a subset of those who have applied under partition α, for every t.*

Proof. For each student $i \in I$ weakly more schools at the low-tail of preferences for each group of schools are truncated at the end of the first iteration $(l-1)$ under partition β than under partition α. Consider a school $s \in S^k$ under partition β and suppose that $s \in \tilde{S}^m$ under partition α. Note that $\tilde{S}^m \subseteq S^k$. Consider the first step of the DA in the second iteration l. Each student applies to his/her top school in each school group under partition β. If a school is student i's top choice under partition β for some group of schools, then it must be his/her top choice school under partition α for some group of school under partition α, as well. Then, it follows that $I^1_\beta(s) \subseteq I^1_\alpha(s)$ where $I^1_\beta(s)$ and $I^1_\alpha(s)$ are the sets of students who applied to school s in the first step of the DA in iteration l.

Suppose that up to Step t, we have $I^x_\beta(s) \subseteq I^x_\alpha(s)$ for every $x = 1, ..., t-1$ and for every $s \in S$, where $I^x_\beta(s)$ and $I^x_\alpha(s)$ are the sets of students who have applied to school $s$at or before Step x. Now, we need to show that $I^t_\beta(s) \subseteq I^t_\alpha(s)$.

Let i be a student who has applied to school s at Step t under partition β, i.e., $i \in I^t_\beta(s) \setminus I^{t-1}_\beta(s)$. We claim that $i \in I^t_\alpha(s)$. If s is the highest-ranked school among $\tilde{S}^m$ under partition α with respect to her truncated preferences at the end of Step $(t-1)$ then $i \in I^1_\alpha(s)$ which implies that $i \in I^t_\alpha(s)$. Otherwise, if s is not the highest-ranked school with respect to her truncated preferences by the end of iteration $(l-1)$, then there exists another school $\tilde{s} \in \tilde{S}^m$ which is ranked right above s among schools in $\tilde{S}^m$. Since $i \in I^t_\beta(s) \setminus I^{t-1}_\beta(s)$ school $\tilde{s}$ must have rejected student i at Step $(t-1)$ or before under partition β, that is $i \in I^{t-1}_\beta(\tilde{s})$ and $i \notin C^{\tilde{s}}(I^{t-1}_\beta(\tilde{s}); q_{\tilde{s}})$. By the inductive assumption, we have $i \in I^{t-1}_\alpha(\tilde{s})$ and by responsiveness we have $i \notin C^{\tilde{s}}(I^{t-1}_\alpha(\tilde{s}); q_{\tilde{s}})$. Therefore i applies to school s at Step t or before under partition α, so $i \in I^t_\alpha(s)$.

We have shown that every student in $I^t_\beta(s) \setminus I^{t-1}_\beta(s)$ is also in $I^t_\alpha(s)$. This and mathematical induction hypothesis that $I^{t-1}_\beta(s) \subseteq I^{t-1}_\alpha(s)$ together imply $I^t_\beta(s) \subseteq I^t_\alpha(s)$. $\square$

Proof. (Theorem 3): Let β be a coarser school partition than α. In Theorem 1 we already show $F^1(P;\beta)RF^1(P;\alpha)$ for a given $P \in \mathcal{P}^{|I|}$. Then, by Lemma 2, for every student any school which survives truncation by the end of iteration 1 under partition β also survives truncation by the end of iteration 1 under partition α. Then, by Lemma 3, the set of students who has applied to a school at or before Step t of the SPDAA under partition β is a subset of those who have applied under partition α, for every t. This, implies that $F^2(P;\beta)RF^2(P;\alpha)$. By Lemma 2, for every student any school which survives truncation by the end of iteration 2 under partition β also survives truncation by the end of iteration 2 under partition α. With a similar argument as before, we have $F^3(P;\beta)RF^3(P;\alpha)$. Proceeding this way, we reach $F^l(P;\beta)RF^l(P;\alpha)$ for a given preference profile $P \in \mathcal{P}$. $\square$

Proof. (Theorem 4): Suppose that school partitions α and β are such that β is coarser than α. By *Manjunath and Turhan (2016)*, for any preference profile P , $F(P;\alpha)$ and $F(P;\beta)$ are both stable matchings. By Theorem 3 in this paper, for every preference profile P and for every student $i \in I$ we have $F_i(P;\beta)R_iF(P;\alpha)$. Then, by either Lemma 1 of *Pathak and Sönmez (2013) or* Theorem 1 of *Chen, Egesdal, Pycia and Yenmez (2016)* we have that for every student $i \in I$, mechanism $F(\cdot;\alpha)$ is as manipulable as mechanism $F(\cdot;\beta)$. $\square$

Proof. (Theorem 5): Formal statement of Theorem 5: Fix a school partition α and consider the mechanism $F^l(\cdot;\alpha)$ for some $l > 1$. Fix preference profile P. For some report of student i, $\tilde{P}_i$, suppose that $F^l_i((\tilde{P}_i, P_{-i});\alpha) = \mu$. Then, there exists a dropping strategy $P^{'}_i$ such that $F^l((P^{'}_i, P_{-i});\alpha)R_i\mu$.

Given a school partition α and a preference profile P, suppose that for a report $\tilde{P}_i$ of student i we have such that $F^l((\tilde{P}_i, P_{-i});\alpha) = \mu$. Call $\mu(i) = s$ and $s \in S^k$ for some k under partition α. Construct $P^{'}_i$ by listing only school s acceptable and reporting every other school in P_i as unacceptable. Let $P^{'} = (P^{'}_i, P_{-i})$. When student i reports $P^{'}_i$ she does not apply any school in school groups other than group k. In the first iteration of the SOSM, by the *population monotonicity*[4] of student proposing deferred acceptance algorithm (Ehlers and Klaus, 2016), every student gets weakly better assignment in school groups other than S^k. Moreover, since student proposing deferred acceptance algorithm is *IR monotonic*[5] every student gets weakly better assignment in school group S^k because student i lists only school s as acceptable. If student i receives school s when he reports $\tilde{P}_i$ in the first iteration of the SOSM, then when he reports $P_i^{'}$ she must receive it in the first iteration, as well. If she receives school s in the second iteration of the SOSM when she reports $\tilde{P}_i$, she receives it in the second iteration of the SOSM when she reports $P_i^{'}$ because after the truncation at the end of the first iteration of the SOSM weakly less number of student will apply schools in group S^k. This holds for any $k = 1, ..., l$ where k is the number of iteration at which student i receives school s when she reports $\tilde{P}_i$. Because for any k the set of students who apply to school s in the course of the student proposing DA when i reports $P_i^{'}$ is a subset of the set of students who apply to school s when i reports $\tilde{P}_i$.

[4] *This property was first introduced by Thomson (1983) in a different framework.*

[5] *For IR monotonicity property see Kojima and Manea (2010).*

□

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