Soc Choice Welf (2018) 50:147–170
https://doi.org/10.1007/s00355-017-1078-0

ORIGINAL PAPER

A simple budget-balanced mechanism

Debasis Mishra[1] · **Tridib Sharma**[2]

Received: 10 November 2016 / Accepted: 27 August 2017 / Published online: 4 September 2017

Abstract In the private values single object auction model, we construct a *satisfactory* mechanism—a dominant strategy incentive compatible and budget-balanced mechanism satisfying *equal treatment of equals*. Our mechanism allocates the object with positive probability to *only* those agents who have the highest value and satisfies ex-post individual rationality. This probability is at least $(1 - \frac{2}{n})$, where n is the number of agents. Hence, our mechanism converges to efficiency at a linear rate as the number of agents grow. Our mechanism has a simple interpretation: a fixed allocation probability is allocated using a second-price Vickrey auction whose revenue is redistributed among all the agents in a simple way. We show that our mechanism maximizes utilitarian welfare among all satisfactory mechanisms that allocate the object *only* to the highest-valued agents.

1 Introduction

This paper considers the problem of allocating a unit of resource among a set of agents who have private valuation for it. Transfers are allowed but preferences over transfers

The authors are grateful to an anonymous referee whose detailed comments has improved the paper significantly. The authors also thank Bhaskar Dutta, Herve Moulin, Anish Sarkar, and seminar participants at Indian Statistical Institute for their comments.

✉ Debasis Mishra
dmishra@isid.ac.in; dmishra@gmail.com

Tridib Sharma
sharma@itam.mx

1 Indian Statistical Institute, Delhi, India

2 CIE, ITAM, Mexico City, Mexico

are quasilinear. Furthermore, transfers have to balance. Examples of such problems include: allocating a bequest among claimants, deciding on a venue of a public good (hospital) among various municipalities, sharing a unit of time on a supercomputer owned jointly by various firms etc.

Efficiency in this problem requires that the agent with the highest valuation be given the entire resource. We follow a mechanism design approach to construct a new dominant strategy incentive compatible (DSIC), budget-balanced, and *nearly* efficient mechanism for this problem. The mechanism design literature on this topic centers around an impossibility result of Green and Laffont (1979): there is no efficient, DSIC, and budget-balanced mechanism. This paper presents a new avenue for escaping this impossibility result by *burning probabilities*.

Relax efficiency by burning probabilities We describe a DSIC, budget-balanced, and individually rational mechanism that *only* allocates probabilities to the highest-valued agent(s) and burns (wastes) the remaining probabilities. With $n \geq 3$ agents and at a generic valuation profile $v_1 > v_2 > \cdots > v_n$, our mechanism allocates the object to agent 1 (highest valuation agent) with probability $(1-\frac{2}{n})+\frac{2}{n}\frac{v_3}{v_2}$. The mechanism can be simply stated as: a second-price auction of this probability followed by a redistribution of the revenue of the second-price auction among all the agents, where agents 1 and 2 receive an amount $\frac{v_3}{n}$, each and every other agent receives an amount $\frac{v_2}{n}$. Such a redistribution is crucial to maintain incentives. Notice that the mechanism converges to efficiency at a linear rate. Our mechanism can be thought to be an answer to the following question:

> What allocation probability can be auctioned using a second-price auction whose revenue can be redistributed among all the agents?

By the Green–Laffont impossibility result, this allocation probability is strictly less than 1, and our mechanism shows that it is larger than $(1 - \frac{2}{n})$. We show that in the class of all mechanisms that allocate the object to only the highest valued agent, our mechanism is *welfare-undominated*, i.e., every mechanism in this class gives less welfare at *some* valuation profile. Specifically, our mechanism maximizes the utilitarian welfare among all DSIC and budget-balanced mechanisms that allocate the object *only* to the highest-valued agents and satisfy a mild fairness property called the *equal treatment of equals*.

We now discuss some of the other attempts to escape the Green–Laffont impossibility theorem and argue how they compare to burning probabilities.

Relax solution concept Cramton et al. (1987) show that there is an efficient, Bayesian incentive compatible, budget-balanced, and individually rational mechanism for this problem.[1] Hence, the Green–Laffont impossibility can be completely overcome by relaxing the solution concept to Bayesian incentive compatibility. We also point out that d'Aspremont and Gérard-Varet (1979) and Arrow (1979) construct mechanisms (now called the dAGV mechanisms), which are efficient, Bayesian incentive

[1] They consider a more general problem with property rights. In our problem, there are no property rights. We can assign equal property rights to all the agents and apply their result.

compatible, and budget-balanced. But the dAGV mechanisms are not individually rational.

The advantage of a DSIC mechanism is that it is prior-free and more robust to strategic manipulation. This is probably the reason that a long literature exists investigating the possibility and impossibility boundaries of DSIC, efficient, and budget-balanced mechanisms—see Hurwicz and Walker (1990), Laffont and Maskin (1980), Green and Laffont (1979), Walker (1980). Our mechanism adds to this literature and provides a new reason to look at DSIC mechanisms.

Relax budget-balance by burning money Another way of overcoming the Green–Laffont impossibility result is to relax the budget-balance constraint. Recent papers follow this approach by relaxing budget-balance to a *no-deficit* condition (i.e., the designer can only earn revenue). Their objective is to redistribute as much revenue as possible from an efficient and DSIC mechanism—Cavallo (2006), Guo and Conitzer (2009) and Moulin (2009, 2010) are notable contributions. By the well-known revenue equivalence results, the only class of efficient and DSIC mechanisms are Groves mechanisms (Holmström 1979). In Guo and Conitzer (2009) and Moulin (2009), they propose Groves mechanisms that redistribute a large fraction of revenue as number of agents grow—unlike the mechanism in Cavallo (2006), these mechanisms are complicated to describe. The main difference from this literature to ours is that budget-balance is a necessary constraint for us, and we are interested in exploring the limitations of imposing DSIC and budget-balance as constraints.

Relax efficiency by giving to others If we burn money, we need not relax efficiency, and we can restrict attention to the Groves class of mechanisms. On the other hand, we may relax efficiency and search within the class of all DSIC and budget-balanced mechanisms. In Long et al. (2017), we describe a class of such mechanisms that we call ranking mechanisms. We further showed that it includes a mechanism which asymptotically converges to efficiency at an exponential rate as the number of agents grow.

Ranking mechanisms include a simple mechanism proposed by Green and Laffont (1979), called the *residual claimant* mechanism. In that mechanism, an agent is uniformly randomly picked to be a residual claimant, and a Vickrey auction is held among the remaining agents. The revenue from the Vickrey auction is given to the residual claimant. This mechanism is DSIC and budget-balanced. It allocates the object to the highest valued agent with probability $(1 - \frac{1}{n})$, where n is the number of agents, and the remaining probability goes to the second highest valued agent.

Relaxing efficiency takes one out of the comfortable class of Groves mechanisms—this means, one needs to worry about both the allocation rule and payment rule. This is the reason we see less work on non-efficient, DSIC, and budget-balanced mechanisms. Besides Long et al. (2017), papers by Hashimoto (2015) and Guo et al. (2011) discuss variants of the Green–Laffont mechanism and its properties. These mechanisms are very close to the Green–Laffont mechanism and differ from our mechanism significantly. Sprumont (2013) characterizes the class of DSIC, individually rational, deficit-free, and envy-free mechanisms. But he does not impose budget-balance. Drexl and Kleiner (2015) investigate expected welfare maximizing DSIC and budget-balanced mechanisms but only consider the case of two agents. Nath and Sandholm

(2016) look at a more general mechanism design problem than ours but restrict attention to mostly deterministic mechanisms. Their main result says that deterministic mechanisms are like Green–Laffont mechanisms but without randomization. With randomization, they give some *approximation* guarantees using Green–Laffont type mechanisms.

In the papers described above, efficiency is relaxed by allocating the object with positive probability to agents who do not have the highest value—the Green–Laffont (GL) mechanism allocates the object to the second highest valued agent with $\frac{1}{n}$ probability and the mechanism in Long et al. (2017) allocate the object to almost $\frac{n}{2}$ agents with positive probability.

There are reasons to worry about mechanisms which allocates the object to non-highest valued agent. One clear reason is that whenever the object is not assigned to the highest valued agent, it can be resold to the highest valued agent ex-post. Though we do not model resale formally (for instance, as in Krishna 2009), such resale opportunities will destroy the incentives of the original mechanism.[2]

From a practical standpoint, this may lead to unpleasant situations sometimes. Consider a scenario where the highest valued agent has valuation 1 million and the second highest valued agent has valuation close to zero. Both the GL mechanism and the mechanisms in Long et al. (2017) allocates the object with positive probability to the second highest valued agent. Giving the object with positive probability to really low-valued agents when a high-valued agent is present may be problematic in certain practical settings. For instance, allocation of spectrum licenses using "first-cum-first-serve" policy led to huge controversies in Indian 2G spectrum allocation.[3] Besides corruption, the Supreme Court of India termed such an allocation as "arbitrary"—probably, hinting that higher valued bidders were not allocated spectrum. Such allocations also led to wide-spread resale of spectrum.[4]

This motivates us to explore a new direction for overcoming the Green–Laffont impossibility result. Compared to the mechanism in Long et al. (2017), our mechanism does not converge to efficiency at an *exponential* rate. However, unlike their mechanism, this mechanism is simpler to describe and *only* allocates the object to the highest valued agents.

Though we show that our mechanism maximizes (ex-post) utilitarian welfare among all DSIC and budget-balanced mechanisms that satisfy equal treatment of equals and allocate only to the highest-valued agents, the welfare comparison between our mechanism and the Green–Laffont mechanism is ambiguous if we do it profile-by-profile. However, we show that if values of each agent is uniformly distributed in [0, 1], then the expected welfare of our mechanism is *less* than that of the Green–Laffont mechanism, but the difference in expected welfare approaches zero at $\frac{1}{n^2}$ rate, where n is the number of agents.

[2] We are grateful to an anonymous referee for suggesting this.

[3] https://en.wikipedia.org/wiki/2G_spectrum_scam.

[4] See a news article on this here: http://timesofindia.indiatimes.com/india/2G-scam-SC-scraps-122-licences-granted-under-Rajas-tenure-trial-court-to-decide-on-Chidambarams-role/articleshow/11725097.cms?referral=PM.

2 The model

We consider the standard single object independent private values model with $N = \{1, \ldots, n\}$ as the set of agents. Throughout, we assume that $n \geq 3$. Each agent $i \in N$ has a valuation v_i for the object. If he is given the object with probability α_i, and he pays p_i for it, then his net utility is $\alpha_i v_i - p_i$. The set of all valuations for any agent is given by $V \equiv [0, \beta]$, where $\beta \in \mathbb{R}$. A valuation profile will be denoted by $\mathbf{v} \equiv (v_1, \ldots, v_n)$.

An **allocation rule** is a map $f : V^n \to [0, 1]^n$, where we denote by $f_i(\mathbf{v})$ the probability of agent i getting the object at valuation profile $\mathbf{v}$. We assume that at all $\mathbf{v} \in V^n$, $\sum_{i \in N} f_i(\mathbf{v}) \leq 1$.

A **payment rule** of agent i is a map $p_i : V^n \to \mathbb{R}$. A collection of payment rules of all the agents will be denoted by $\mathbf{p} \equiv (p_1, \ldots, p_n)$. A **mechanism** is a pair $(f, \mathbf{p})$. We require our mechanism to satisfy the following three properties—all these properties and the individual rationality notion we use are *ex-post* properties, and we suppress the qualifier ex-post throughout the paper.

- A mechanism $(f, \mathbf{p})$ is **dominant strategy incentive compatible (DSIC)** if for every $i \in N$, for every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$, we have

$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v_i', v_{-i}) - p_i(v_i', v_{-i}).$$

- A mechanism $(f, \mathbf{p})$ is **budget-balanced (BB)** if for every $\mathbf{v} \in V^n$, we have

$$\sum_{i \in N} p_i(\mathbf{v}) = 0.$$

- A mechanism $(f, \mathbf{p})$ satisfies **equal treatment of equals (ETE)** if for every $\mathbf{v} \in V^n$ and for every $i, j \in N$ with $v_i = v_j$, we have

$$f_i(\mathbf{v}) = f_j(\mathbf{v}), \quad p_i(\mathbf{v}) = p_j(\mathbf{v}).$$

We call a mechanism **satisfactory** if it is DSIC, BB, and ETE and call an allocation rule f **satisfactorily implementable** if there exists $\mathbf{p}$ such that $(f, \mathbf{p})$ is satisfactory. ETE allows us to consider a mild notion of fairness in our mechanism. It also explicitly rules out *dictatorial* mechanisms, where a dictator agent is given the object for free at all valuation profiles. We are interested in finding satisfactory mechanisms that are almost efficient in the following sense.

At any valuation profile $\mathbf{v}$, denote by $\mathbf{v}[k]$ the set of agents who have the k-th highest valuation at $\mathbf{v}$. More formally,

$$\mathbf{v}[1] := \{i \in N : v_i \geq v_j \ \forall \ j \in N\}.$$

Having defined $\mathbf{v}[k-1]$, we recursively define $\mathbf{v}[k]$ as

$$\mathbf{v}[k] := \left\{ i \in N \setminus \left(\cup_{k'=1}^{k-1} \mathbf{v}[k'] \right) : v_i \geq v_j \ \forall \ j \in N \setminus \left(\cup_{k'=1}^{k-1} \mathbf{v}[k'] \right) \right\}.$$

Definition 1 An allocation rule f is **efficient** at $\mathbf{v}$ if

$$\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1.$$

An allocation rule f is efficient if it is efficient at all $\mathbf{v} \in V^n$. A mechanism $(f, \mathbf{p})$ is efficient if f is efficient.

The efficiency of a BB mechanism is equivalent to maximizing the total welfare of agents at every profile of valuations. To see this, note that the total welfare of agents at a valuation profile $\mathbf{v}$ from a mechanism $(f, \mathbf{p})$ is

$$\sum_{i \in N} [v_i f_i(\mathbf{v}) - p_i(\mathbf{v})] = \sum_{i \in N} v_i f_i(\mathbf{v}),$$

where the second equality followed from BB. This is clearly maximized by assigning the object to the highest valued agents.

3 A top-only satisfactory mechanism

We now define our mechanism. Informally, the mechanism can be described in very simple terms as follows.

1. Agents are asked to report their values, and suppose the reported values are $v_1 > v_2 > \cdots > v_n$—we consider reported values to be strictly ordered for simplicity.
2. Probability $\pi(v_2, v_3) = (1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}$ is auctioned using a second-price auction. In particular,
   (a) Agent 1 wins the probability $\pi(v_2, v_3)$.
   (b) Agent 1 pays $v_2 \pi(v_2, v_3) \equiv (1 - \frac{2}{n})v_2 + \frac{2}{n} v_3$.
3. To maintain budget-balance, the generated revenue from the second-price auction, $v_2 \pi(v_2, v_3)$, is redistributed among agents as follows:
   (a) Agents 1 and 2 receive an amount $\frac{v_3}{n}$ each.
   (b) Each agent j, where $j > 2$, receives an amount $\frac{v_2}{n}$.

Before formally defining the mechanism, we comment on some obvious properties of the mechanism. The probability auctioned in the mechanism depends on the (reported) values of second and third highest valued agents. Loosely, this cannot distort the incentives in the auction because all the allocation probabilities only go to the highest valued agent. Further, the redistribution amount of each agent does not depend on his own reported valuation, and hence, maintains incentive compatibility. This makes the overall mechanism DSIC. It is clearly budget-balanced. By breaking the ties carefully, we make it satisfy ETE. Finally, each agent gets non-negative payoff in the mechanism, ensuring individual rationality. Also, by definition, only the highest valued agent gets the object with positive probability.

We now define the mechanism carefully to handle ties in reported values.

Definition 2 Mechanism $M^* \equiv (f^*, \mathbf{p}^*)$ is defined as follows. The allocation rule f^* is defined as: for every $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n$, we have

$$f_i^*(\mathbf{v}) := \begin{cases} \frac{1}{|\mathbf{v}[1]|}\Big[\big(1 - \frac{2}{n}\big) + \big(\frac{2}{n}\big)\frac{v_3}{v_2}\Big] & \text{if } i \in \mathbf{v}[1] \\ 0 & \text{otherwise} \end{cases}$$

where $\frac{0}{0}$ is assumed to be 1. The payment of each agent $i \in N$ is given by

$$p_i^*(\mathbf{v}) := p_i^*(0, v_{-i}) + v_i f_i^*(\mathbf{v}) - \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i,$$

where $p_i^*(0, v_{-i})$ is defined as

$$p_i^*(0, v_{-i}) = \begin{cases} -\frac{v_3}{n} & \text{if } i \in \{1, 2\} \\ -\frac{v_2}{n} & \text{otherwise.} \end{cases}$$

In the definition above, we can write $p_i^*(0, v_{-i})$ equal to the second highest value in v_{-i}, which makes it clear that it is independent of the value of agent i.

Though the formal definition involves defining payments using a Myersonian formula, it coincides with our informal description for the generic case when $v_1 > v_2 > v_3 > \cdots > v_n$. To see this, note that in this case, $f_1(\mathbf{v}) = \pi(v_2, v_3) = (1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}$ and $f_i(\mathbf{v}) = 0$ for all $i > 1$. Further, $f_1(x_1, v_{-1}) = \pi(v_2, v_3)$ for all $x_1 \in (v_2, v_1]$ and $f_1(x_1, v_{-1}) = 0$ for all $x_1 < v_2$. Finally, $f_i(x_i, v_{-i}) = 0$ for all $x_i \leq v_i$ for all $i \neq 1$. These observations imply that the payment defined using the Myersonian formula in the above description coincides with the payments in the informal description.

Tie-Breaking Tie-breaking in our mechanism is done in a symmetric way. We illustrate this with an example. Suppose $N = \{1, 2, 3\}$. There are three possible ties that can happen, and we describe our mechanism in each of the cases.

1. Suppose $v_1 = v_2 = v_3$. Then the object is given to each agent with equal probability and no probability is burnt. Further for every i, $p_i^*(0, v_{-i}) = -\frac{v_1}{3} = -\frac{v_2}{3} = -\frac{v_3}{3}$. Hence, using the revenue equivalence formula, we get that the payment of every agent is zero. So, agents are distributed equal share of the object for free.
2. Suppose $v_1 = v_2 > v_3$. Then, the object is given with equal probability to agents 1 and 2, but some probability is burnt. In this case, agent 3 *receives* a payment of $\frac{v_2}{3}$. For every $i \in \{1, 2\}$, using the fact that $p_i^*(0, v_{-i}) = -\frac{v_3}{3}$ and the revenue equivalence formula, we get $p_i^*(v_1, v_2, v_3) = \frac{v_i}{6}$.
   These amounts correspond to a uniform randomization over two asymmetric Vickrey auction. In the first auction, the tie is broken in favor of agent 1, and the other, it is broken in favor of agent 2. In each auction, a probability of $\frac{1}{3} + \frac{2}{3}\frac{v_3}{v_2}$ is auctioned—in one auction, the winner is agent 1 and the other the winner is agent 2. In either case, the winner pays an amount equal to $\frac{v_2}{3} + \frac{2v_3}{3}$. This amount is shared between the agents as follows: agents 1 and 2 get $\frac{v_3}{3}$ each and agent 3 gets $\frac{v_2}{3}$. Uniform randomization over these two auctions exactly give us our mechanism, and generates a mechanism satisfying ETE.

3. Suppose $v_1 > v_2 = v_3$. Then, the object is given with probability 1 to agent 1. In this case, agent 2 receives a payment equal to $\frac{v_3}{3}$ and agent 3 receives a payment equal to $\frac{v_2}{3} = \frac{v_3}{3}$. Hence, payment of agent 1 is $\frac{2v_3}{3}$.
   This amount exactly corresponds to the fact that a Vickrey auction of the entire object is conducted. This generates a revenue of v_2, which is distributed equally among all the agents, including agent 1 (the winner).

3.1 The result

In this section, we state the main result of the paper. Before describing the main result, we introduce some notation. For satisfactory mechanism $M \equiv (f, \mathbf{p})$, let $W(\mathbf{v}; M)$ be the welfare generated at a valuation profile $\mathbf{v}$ by this mechanism:

$$W(\mathbf{v}; M) := \sum_{i \in N} [v_i f_i(\mathbf{v}) - p_i(\mathbf{v})] = \sum_{i \in N} v_i f_i(\mathbf{v}),$$

where the second equality follows from budget-balance.

Definition 3 An allocation rule f is **top-only** if at every valuation profile $\mathbf{v}$, $f_i(\mathbf{v}) = 0$ if $i \notin \mathbf{v}[1]$. A mechanism $M \equiv (f, \mathbf{p})$ is a top-only mechanism if f is a top-only allocation rule.

The next definition is about the participation constraint of a mechanism.

Definition 4 A mechanism $M \equiv (f, \mathbf{p})$ satisfies **individual rationality** if for every $\mathbf{v}$ and every $i \in N$, we have

$$v_i f_i(\mathbf{v}) - p_i(\mathbf{v}) \geq 0.$$

Finally, we make the definition of maximization of utilitarian welfare precise.

Definition 5 For a class of satisfactory mechanisms $\mathcal{M}$, a satisfactory mechanism $M \in \mathcal{M}$ **maximizes utilitarian welfare** in $\mathcal{M}$ if there exists no other satisfactory mechanism $M' \in \mathcal{M}$ such that

$$W(\mathbf{v}; M') \geq W(\mathbf{v}; M) \qquad \forall\, \mathbf{v},$$

with strict inequality holding for some $\mathbf{v}$.

So, maximizing utilitarian welfare gives us a mechanism in the Pareto frontier of $\mathcal{M}$. We are now ready to state the main result of the paper.

Theorem 1 *The mechanism $M^* \equiv (f^*, \mathbf{p}^*)$ is a top-only satisfactory mechanism satisfying individual rationality. Further, it maximizes utilitarian welfare in the class of all top-only satisfactory mechanisms.*

The proof of this theorem is in the Appendix.

4 Welfare comparison

In this section, we compare the welfare properties of our mechanism with some existing DSIC and *almost* efficient mechanisms. The comparisons are done in three subsections and we provide a brief preview of these results before formally stating them.

In Sect. 4.1, we attempt to do an ex-ante welfare comparison of our mechanism with the celebrated Green–Laffont (GL) mechanism. For uniform iid draws of values, the GL mechanism generates more expected welfare than our mechanism. In Sect. 4.2, we provide the following foundation for the GL mechanism: there is a modification of the GL mechanism (GL mechanism is only modified at measure zero valuation profiles) that maximizes utilitarian welfare among all satisfactory mechanisms. Hence, for uniform iid draws of values, this modified GL mechanism also generates more expected welfare than our mechanism. However, the computations for other distributions turn out to be intractable. Even for uniform iid draws of values, the difference in expected welfare between the GL (or modified GL) mechanism and our mechanism goes to zero at a quadratic rate as we increase the number of agents. Further, there is a *positive* measure of valuation profiles where our mechanism generates more (ex-post) welfare than the GL mechanism and the modified GL mechanism.

Finally, in Sect. 4.3, we show that if we are willing to relax budget-balance to *no deficit* there is a family of modifications of Vickrey auction that generates more (ex-post) welfare than our mechanism.

4.1 Welfare comparison with the GL mechanism

The literature has exclusively dealt with DSIC and budget-balanced mechanisms that never burn probabilities but allocate the object with positive probability to non-highest-valued agents. One simple mechanism that achieves this is the GL mechanism.

We discuss efficiency of the GL mechanism and our mechanism. To remind the readers, the GL mechanism is a DSIC and budget-balanced mechanism that allocates the object to the highest valued agent with probability $(1-\frac{1}{n})$ and to the second highest valued agent with probability $\frac{1}{n}$. Hence, the *total welfare* (sum of utilities of agents) at a valuation profile $\mathbf{v}$ with $v_1 \geq v_2 \geq \cdots \geq v_n$ in the GL mechanism is

$$\left(1-\frac{1}{n}\right)v_1+\frac{1}{n}v_2$$

and in our mechanism M^*

$$\left(1-\frac{2}{n}\right)v_1+\frac{2}{n}\frac{v_3}{v_2}v_1.$$

Hence, our mechanism generates more welfare than the GL mechanism if and only if $\frac{2}{n}\frac{v_1 v_3}{v_2} \geq \frac{1}{n}(v_1+v_2)$. This is equivalent to requiring

$$2\frac{v_3}{v_2} \geq 1+\frac{v_2}{v_1}.$$

Notice that if valuations are drawn from some compact interval $[0, \beta]$, where $\beta > 0$, the set of profiles where this condition is satisfied has positive Lebesgue measure. In particular, from an ex-ante perspective, it is not clear which of these two simple mechanisms can give higher expected welfare—it will depend on the prior distribution being considered.

Such a comparison seems difficult to do for general value distributions. However, for uniform distribution, the expected welfare terms become tractable. So, we assume that values of all the agents are uniformly drawn from [0, 1].

We use the following important fact from statistics for this. Let $U_{(1):n} \leq U_{(2):n} \leq \cdots \leq U_{(n):n}$ be the **order statistics** of n IID random variables, where $U_{(j):n}$ is the j-th **lowest** of the ordered sample.

Fact 1 (Malmquist 1950; Reiss 2012) *The following are true about ratios of order statistics.*

1. *The ratios* $\frac{U_{(1):n}}{U_{(2):n}}, \ldots, \frac{U_{(n-1):n}}{U_{(n):n}}, \frac{U_{(n):n}}{U_{(n+1):n}}$, *where* $U_{(n+1):n} \equiv 1$, *are independent random variables.*
2. *The random variable* $\frac{U_{(k):n}}{U_{(k+1):n}}$ *is distributed the same as* $U_{(k):k}$.

Using the expressions for total welfare of both the mechanisms, we now compute the expected total welfares. The computations are fairly straightforward for the GL mechanism (below, we use $\mathbb{E}[\cdot]$ to denote the expectation operator):

$$\begin{aligned} W^{GL} &= \mathbb{E}\left[\left(1-\frac{1}{n}\right)U_{(n):n} + \frac{1}{n}U_{(n-1):n}\right] \\ &= \left(1-\frac{1}{n}\right)\frac{n}{(n+1)} + \frac{1}{n}\frac{(n-1)}{(n+1)} \\ &= \frac{(n-1)}{n}. \end{aligned}$$

For computing the expected welfare from our mechanism, we use Fact 1:

$$\begin{aligned} W^{M^*} &= \mathbb{E}\left[\left(1-\frac{2}{n}\right)U_{(n):n} + \frac{2}{n}\frac{U_{(n-2):n}}{U_{(n-1):n}}U_{(n):n}\right] \\ &= \left(1-\frac{2}{n}\right)\frac{n}{(n+1)} + \frac{2}{n}\mathbb{E}\left[\frac{U_{(n-2):n}}{U_{(n-1):n}}\right]\mathbb{E}[U_{(n):n}] \\ &\quad \text{(Using (1) of Fact 1 with the fact that if X and Y are independent, then } \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].\text{)} \\ &= \left(1-\frac{2}{n}\right)\frac{n}{(n+1)} + \frac{2}{n}\frac{(n-2)}{(n-1)}\frac{n}{(n+1)} \quad \text{(Using (2) of Fact 1)} \\ &= \frac{(n-2)}{(n-1)} \end{aligned}$$

Now, we can easily see that,

$$W^{GL} - W^{M^*} = \frac{(n-1)}{n} - \frac{(n-2)}{(n-1)} = \frac{1}{n(n-1)} > 0$$

We document this fact as a proposition. For any functions $a\colon \mathbb{N} \to \mathbb{R}$ and $b\colon \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of positive integers, we write $a(n) \sim b(n)$ to mean that $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$.

Proposition 1 *If values are uniformly distributed in* $[0, 1]$, *then* $W^{GL} > W^{M^*}$, *but*

$$\left(W^{GL} - W^{M^*}\right) \sim \frac{1}{n^2}.$$

Proposition 1 shows that for uniform distribution, our mechanism does worse than the celebrated GL mechanism in terms of ex-ante welfare. But the difference approaches zero at $\frac{1}{n^2}$ rate.[5] Given that both the mechanisms are relatively simple to describe, such asymptotic equivalence of the two mechanisms in terms of ex-ante welfare gives another practical reason to consider our mechanism besides the top-only property. The computations for other distributions turn out to be intractable.

4.2 A foundation for the GL mechanism

A natural question to ask is whether a counterpart of our main result can be established if we allow for top two highest valued agents to get the object. A candidate mechanism in this class is the GL mechanism. We show that the GL mechanism can be *welfare-dominated* in the sense of Theorem 1. In particular, we will show that a *simple/non-generic* modification of the GL mechanism welfare-dominates the GL mechanism. Further, this modified GL mechanism maximizes utilitarian welfare in the class of *all* satisfactory mechanism.

We now propose a modification of the Green–Laffont mechanism that reduces the set of valuation profiles where inefficiency occurs. To remind readers, the GL mechanism is efficient whenever there are more than one agent with highest valuation. The modification we propose modifies the GL mechanism at valuation profiles where there is a *unique* highest-valued agent but *more than one* second-highest valued agent—in these valuation profiles, the modified GL will allocate the object with **probability 1** to the highest valued agent (the GL mechanism allocates the object with probability $1 - \frac{1}{n}$ to the highest valued agent and probability $\frac{1}{n}$ is equally shared between all second-highest valued agents). The payments are modified accordingly to maintain incentive compatibility. This is the only difference between the GL and the modified GL mechanism. We now formally define the modified GL mechanism.

Notation At a valuation profile $\mathbf{v}$, we use $v_{(j)}$ to denote the valuation of agents in $\mathbf{v}[j]$.

Definition 6 The mechanism $M^{G'} \equiv (f^{G'}, \mathbf{p}^{G'})$ is the **modified Green–Laffont (MGL) mechanism** if at every profile of valuations $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n$

[5] This means that for $n = 5$, the difference in expected welfare is 0.05.

- if $|\mathbf{v}[1]| = 1$ **and** $|\mathbf{v}[2]| = 1$ we have

$$f_j^{G'}(\mathbf{v}) = \begin{cases} 1 - \frac{1}{n} & \text{if } j \in \mathbf{v}[1] \\ \frac{1}{n} & \text{if } j \in \mathbf{v}[2] \\ 0 & \text{otherwise} \end{cases}$$

  and

$$p_j^{G'}(\mathbf{v}) = \begin{cases} v_2\left(1 - \frac{2}{n}\right) & \text{if } j \in \mathbf{v}[1] \\ 0 & \text{if } j \in \mathbf{v}[2] \\ -\frac{v_2}{n} & \text{otherwise.} \end{cases}$$

- if $|\mathbf{v}[1]| = 1$ **and** $|\mathbf{v}[2]| > 1$, then $f^{G'}$ is efficient at $\mathbf{v}$ (i.e., $f_j^{G'}(\mathbf{v}) = 1$ if $j \in \mathbf{v}[1]$ and $f_j^{G'}(\mathbf{v}) = 0$ for all $j \notin \mathbf{v}[1]$) and

$$p_j^{G'}(\mathbf{v}) = \begin{cases} v_2\left(1 - \frac{1}{n}\right) & \text{if } j \in \mathbf{v}[1] \\ -\frac{v_2}{n} & \text{otherwise.} \end{cases}$$

- else $f^{G'}$ is efficient at $\mathbf{v}$ with $f_j^{G'}(\mathbf{v}) = f_k^{G'}(\mathbf{v})$ for all $j, k \in \mathbf{v}[1]$ and

$$p_j^{G'}(\mathbf{v}) = \begin{cases} \frac{v_1}{|\mathbf{v}[1]|} - \frac{v_1}{n} & \text{if } j \in \mathbf{v}[1] \\ -\frac{v_1}{n} & \text{if } j \notin \mathbf{v}[1]. \end{cases}$$

The allocation rule used in the MGL mechanism will be called the MGL allocation rule.

Note that if $\{i\} = \mathbf{v}[1]$ and $|\mathbf{v}[2]| > 1$, then $f_i^{G'}(\mathbf{v}) = 1$—these are the only valuation profiles where the MGL mechanism achieves efficiency but the Green–Laffont mechanism is not efficient. Thus, clearly, the modified GL generates more utilitarian welfare than the GL mechanism. We are now ready to state the main result of this section.

Theorem 2 *The MGL mechanism $M^{G'}$ is satisfactory. Further, it maximizes utilitarian welfare in the class of all satisfactory mechanisms.*

The proof of this theorem is in the Appendix. We make some remarks about the MGL mechanism and Theorem 2.

1. Independent of our work, Hashimoto (2015) discovers the MGL mechanism. He proves a welfare optimality property of the MGL mechanism which is *weaker* than the utilitarian welfare maximization proved in Theorem 2. According to his criteria, a satisfactory mechanism is *Pareto optimal* if there does not exist another satisfactory mechanism such that *every* agent gets more utility at every valuation profile in the new mechanism. On the other hand, our notion requires the same property but on the aggregate utility of agents. Hence, our Theorem 2 is not implied by the result in Hashimoto (2015).

2. The MGL mechanism only allocates the object to top two highest-valued agents and it does *not* waste any object. Hence, Theorem 2 shows that there is a utilitarian welfare maximizing mechanism that *only* allocates to top two highest-valued agents. On the other hand, we can show such utilitarian welfare maximizing property for our top-only mechanism M^* only inside the class of top-only satisfactory mechanisms. Whether M^* also maximizes utilitarian welfare in the class of *all* satisfactory mechanisms remain an open question.
3. The set of valuation profiles where the GL and the MGL mechanisms differ has measure zero. Hence, by Proposition 1, our top-only mechanism M^* generates less expected welfare than the MGL mechanism if the values are iid draws from uniform distribution. But the difference in expected welfare approaches zero at the rate $\frac{1}{n^2}$.

4.3 A class of no-deficit mechanisms

Now, we return to the issue of burning money instead of burning probabilities to escape the Green–Laffont impossibility. We describe a class of *no-deficit* mechanisms that welfare dominates our mechanism. This essentially hints that burning money may be better than burning probabilities to increase welfare—we are being careful here because we have not explored the entire class of top-only mechanisms. However, we stress here that asymptotically these mechanisms have similar welfare properties. Moreover, money-burning mechanisms are impractical in settings where budget-balance is a hard constraint.

Before describing our new class of mechanisms, we first give a formal definition of no-deficit mechanisms.

Definition 7 A mechanism $(f, \mathbf{p})$ satisfies **no-deficit** if for every $\mathbf{v} \in V^n$, we have

$$\sum_{i \in N} p_i(\mathbf{v}) \geq 0.$$

We use the idea of our mechanism to construct a class of no-deficit mechanism. The extremes of this class is our mechanism and the mechanism by Cavallo (2006). As we go from our mechanism to the Cavallo mechanism inside this class, the utility of every agent increases, achieving the maximum at the Cavallo mechanism. At the same time, as we go from our mechanism to the Cavallo mechanism inside this class, (a) the amount money burning increases and (b) the amount of probability burning decreases.

The class of mechanisms we define are parametrized by $\lambda \in [0, 1]$. We call such a mechanism λ-**Vickrey-Redistribution** mechanism.

1. Agents are asked to report their values, and suppose the reported values are $v_1 > v_2 > \cdots > v_n$—we consider reported values to be strictly ordered for simplicity.
2. Probability

$$\pi^{\lambda}(v_2, v_3) = \lambda \left[\left(1 - \frac{2}{n}\right) + \frac{2}{n} \frac{v_3}{v_2} \right] + (1 - \lambda)$$

is auctioned using a second-price auction. In particular,
   (a) Agent 1 wins the probability $\pi^\lambda(v_2, v_3)$.
   (b) Agent 1 pays $v_2\pi^\lambda(v_2, v_3) \equiv \lambda\big[(1 - \frac{2}{n})v_2 + \frac{2}{n}v_3\big] + (1 - \lambda)v_2$.
3. Part of the generated revenue from the second-price auction, $v_2\pi^\lambda(v_2, v_3)$, is redistributed among agents as follows:
   (a) Agents 1 and 2 receive an amount $\frac{v_3}{n}$ each.
   (b) Each agent j, where $j > 2$, receives an amount $\frac{v_2}{n}$.

The 1-Vickrey-redistribution mechanism is our mechanism and 0-Vickrey-redistribution mechanism is the Cavallo mechanism. In the Cavallo mechanism, a Vickrey auction of the entire unit of resource is conducted. The revenue raised from the auction is redistributed exactly like our auction, but this leaves some surplus, which is burnt.

We can formally break ties in our class of no-deficit mechanisms by maintaining ETE—this can be analogously done to the formal definition of our mechanism M^λ.

Definition 8 The mechanism $M^\lambda \equiv (f^\lambda, \mathbf{p}^\lambda)$ for any $\lambda \in [0, 1]$ is defined as follows. The allocation rule f^λ is defined as: for every $\mathbf{v}$ with $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n$, we have

$$f_i^\lambda(\mathbf{v}) := \begin{cases} \frac{1}{|\mathbf{v}[1]|}\left[\lambda\left(\left(1 - \frac{2}{n}\right) + \left(\frac{2}{n}\right)\frac{v_3}{v_2}\right) + (1 - \lambda)\right] & \text{if } i \in \mathbf{v}[1] \\ 0 & \text{otherwise} \end{cases}$$

where $\frac{0}{0}$ is assumed to be 1. The payment of each agent $i \in N$ is given by

$$p_i^\lambda(\mathbf{v}) := p_i^*(0, v_{-i}) + v_i f_i^\lambda(\mathbf{v}) - \int_0^{v_i} f_i^\lambda(x_i, v_{-i})dx_i,$$

where $p_i^*(0, v_{-i})$ is defined as

$$p_i^*(0, v_{-i}) = \begin{cases} -\frac{v_3}{n} & \text{if } i \in \{1, 2\} \\ -\frac{v_2}{n} & \text{otherwise.} \end{cases}$$

Notice that the redistribution amounts p_i^* remains the same irrespective of the value of λ. The proof that any such mechanism is DSIC follows arguments similar to Theorem 1, and is skipped—it can also be shown using the fact that each mechanism in M^λ is a convex combination of M^* and the Cavallo mechanism. It clearly satisfies individual rationality. The surplus generated by such a λ-Vickrey redistribution mechanism is the following at valuation profile $\mathbf{v}$.

$$\lambda\left[\left(1 - \frac{2}{n}\right)v_2 + \frac{2}{n}v_3\right] + (1 - \lambda)v_2 - \frac{2}{n}v_3 - \left(1 - \frac{2}{n}\right)v_2$$
$$= \frac{2}{n}(1 - \lambda)(v_2 - v_3) \geq 0.$$

Hence, each λ-Vickrey-redistribution mechanism satisfies no-deficit, and for $\lambda = 1$, we have budget-balance. We summarize these conclusions in the following proposition.

Proposition 2 *Every λ-Vickrey redistribution mechanism is DSIC and satisfies individual rationality and no-deficit.*

Fix any λ-Vickrey-redistribution mechanism. At any valuation profile **v** (consider a strict valuation profile $v_1 > v_2 > \cdots > v_n$), the utility of agent j, where $j \notin \{1, 2\}$ is $\frac{v_2}{n}$. The utility of agent 2 is $\frac{v_3}{n}$. The utility of agent 1 is

$$\begin{aligned}(v_1 - v_2)\pi^\lambda(v_2, v_3) + \frac{v_3}{n} &= \lambda(v_1 - v_2)\left[\left(1 - \frac{2}{n}\right) + \frac{2}{n}\frac{v_3}{v_2}\right] \\ &\quad + (1 - \lambda)(v_1 - v_2) + \frac{v_3}{n} \\ &= (v_1 - v_2)\left[\left(1 - \frac{2}{n}\right) + \frac{2}{n}\frac{v_3}{v_2}\right] \\ &\quad + \frac{v_3}{n} + (1 - \lambda)(v_1 - v_2)\frac{2}{n}\left(1 - \frac{v_3}{v_2}\right)\end{aligned}$$

Hence, the utility of agent 1 is strictly increasing with decreasing λ. On the other hand, the utilities of other agents are unchanged. Hence, by reducing λ, we increase the surplus that needs to be burnt but make the highest valued agent better off. This illustrates that the ability to burn some surplus allows one greater flexibility to increase welfare. The budget-balance condition constraints our mechanism, though asymptotically both the mechanisms have similar welfare.

5 Conclusion

We argued that the top-only feature is compelling because it prevents potential resale markets and avoids the pitfall of assigning the object to low-valued agents in the presence of high-valued agents. However, we observed that the GL mechanism generates more expected welfare than our mechanism for uniformly distributed values. Further, a modified GL mechanism maximizes utilitarian welfare over all satisfactory mechanisms. We also observed that there are simpler variants of the Vickrey auction which satisfies *no-deficit*, DSIC, and individual rationality that can generate more welfare than our mechanism. Hence, it is not clear that our mechanism is a clear winner in the class of mechanisms available in this setting. But we believe that it is a valuable addition to this class of mechanisms because of (a) the importance of the top-only property (that we highlight in the Introduction), (b) its simplicity and asymptotic efficiency property, and (c) its asymptotic equivalence to the celebrated GL mechanism.

Besides, our mechanism sheds insights into some technical issues on designing satisfactory mechanisms. First, our mechanism M^* cannot be expressed as a convex combination of deterministic, DSIC, and budget-balanced mechanisms[6]—note that the GL mechanism can be expressed in that form.

Second, our mechanism is a *non-ranking* DSIC and budget-balanced mechanism—Long et al. (2017) define a ranking mechanism as one which allocates a fixed

[6] This follows from the top-only property of our mechanism.

probability π_k to the k-th highest valued agent at every valuation profile. Our mechanism is a non-ranking mechanism because it allocates different probabilities to the highest-valued agent. Ours is the first paper to carefully analyze a non-ranking DSIC and budget-balanced mechanism and establish its optimality and asymptotic properties.

Finally, ours is the first paper to explore the power of a top-only mechanism and illustrate that probability burning may help in partially overcoming the Green–Laffont impossibility result.

Appendix

Proof of Theorem 1

First, we show that M^* is a satisfactory mechanism—it is clearly a top-only mechanism. Fix a valuation profile $\mathbf{v}$ with $v_1 \geq v_2 \geq \cdots \geq v_n$, and observe the following using the definition of $p_i(\mathbf{v})$ for each i:

$$\begin{aligned}
\sum_{i \in N} p_i^*(\mathbf{v}) &= \sum_{i \in N} p_i^*(0, v_{-i}) + \sum_{i \in N} v_i f_i^*(\mathbf{v}) - \sum_{i \in N} \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i \\
&= \sum_{i \in N} p_i^*(0, v_{-i}) + v_1 f_1^*(\mathbf{v}) - \sum_{i \in N} \int_0^{v_i} f_i^*(x_i, v_{-i}) dx_i \quad \text{(by ETE)} \\
&= \sum_{i \in N} p_i^*(0, v_{-i}) + v_1 f_1^*(\mathbf{v}) - (v_1 - v_2) f_1^*(\mathbf{v}) \quad \text{(by definition of } f^*\text{)} \\
&= \sum_{i \in N} p_i^*(0, v_{-i}) + v_2 f_1^*(\mathbf{v}) \\
&= \sum_{i \in N} p_i^*(0, v_{-i}) + v_2 \left(1 - \frac{2}{n}\right) + v_3 \frac{2}{n} \\
&= 0 \quad \text{(by definition of } p_i^*(0, v_{-i}) \text{ for each } i\text{)}.
\end{aligned}$$

This establishes that M^* is budget-balanced. For DSIC, we invoke the characterization of Myerson (1981), which states that an arbitrary mechanism $M \equiv (f, \mathbf{p})$ is DSIC if and only if

1. Monotonicity. for all $i \in N$, for all v_{-i}, and for all v_i, v_i' with $v_i > v_i'$, we have

$$f_i(v_i, v_{-i}) \geq f_i(v_i', v_{-i}). \tag{1}$$

2. Revenue equivalence. for all $i \in N$, for all v_{-i}, and for all v_i, we have

$$p_i(v_i, v_{-i}) = p_i(0, v_{-i}) + v_i f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(x_i, v_{-i}) dx_i. \tag{2}$$

Monotonicity is clearly satisfied by f^* and revenue equivalence is satisfied by $\mathbf{p}^*$ by definition. Hence, M^* is DSIC. Finally, since f^* is symmetric, $\mathbf{p}^*$ is also symmetric by construction. Hence, M^* satisfies ETE. This implies that M^* is a top-only satisfactory mechanism.

For individual rationality, note that for every $i \in N$ and for all $\mathbf{v}$, using revenue equivalence, we have

$$v_i f_i^*(\mathbf{v}) - p_i^*(\mathbf{v}) = \int_0^{v_i} f_i^*(x_i, v_{-i})dx_i - p_i^*(0, v_{-i}) \geq 0,$$

where the inequality follows since $p_i^*(0, v_{-i}) \leq 0$ by definition.

Now, we move to the second part of the proof where we show that our top-only satisfactory mechanism maximizes utilitarian welfare in the class of all top-only satisfactory mechanisms. To do this, we define some additional properties of an allocation rule, which is satisfied by f^*.

Definition 9 An allocation rule f satisfies property

P0. if for every $\mathbf{v}$ with $|\mathbf{v}[1]| = 2$, we have $f_i(\mathbf{v}) = 0$ for all $i \notin \mathbf{v}[1]$.
P1. if for every $\mathbf{v}$ with $|\mathbf{v}[1]| > 2$, we have $\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1$.
P2. if for every $\mathbf{v}$ with $\mathbf{v}[1] = \{k\}$ and $|\mathbf{v}[2]| > 1$, we have $f_k(\mathbf{v}) = 1$.

Notice that f^* satisfies Properties P0, P1, and P2. Before completing the proof of the theorem, we state and prove an important proposition.

Proposition 3 *Suppose* $(f, \mathbf{p})$ *is a satisfactory mechanism and* f *satisfies Properties P0, P1, and P2. Then, for every* $\mathbf{v}$ *with* $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_n$, *we have*

$$\sum_{i \in N} p_i(0, v_{-i}) = -\frac{1}{n}[(n-2)v_2 + 2v_3].$$

Proof We start off by establishing a property of payments.

Lemma 1 *Suppose* $(f, \mathbf{p})$ *is a satisfactory mechanism and* f *satisfies Properties* P0, P1, *and* P2. *For every* $v_{-1} \equiv (v_2, v_3, \ldots, v_n)$ *with* $v_2 \geq v_3 \geq \cdots \geq v_n$, *we have*

$$p_1(0, v_{-1}) = -\frac{v_3}{n}.$$

Proof We do the proof in three steps.

STEP 1. Pick v_{-1} such that $v_2 = v_3 = \theta \geq v_4 \geq \cdots \geq v_n$. Pick a type profile $\mathbf{v} \equiv (v_1, v_{-1})$ such that $v_1 = \theta$. If $\theta = 0$ this is the zero type profile, and by ETE and budget-balance, the claim is true. Hence, suppose that $\theta > 0$. Let $K := |(0, v_{-1})[1]|$. Since $K \geq 2$, we have $|\mathbf{v}[1]| > 2$, and Property P1 implies that $\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1$. Further, consider a type profile (x_1, v_{-1}), where $x_1 < \theta$. Such a type profile also satisfies $|(x_1, v_{-1})[1]| > 1$, and Property P0 and P1 imply that $f_1(x_1, v_{-1}) = 0$.

We now do the proof using induction on K. Using the observations in the previous paragraph along with ETE and revenue equivalence formula, we get for all $i \in \mathbf{v}[1]$,

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) + \frac{1}{K+1}\theta. \tag{3}$$

If $K = n-1$, then $\mathbf{v}[1] = N$, and adding the above inequalities and using ETE and BB, we get

$$p_1(0, v_{-1}) = -\frac{\theta}{n}.$$

Else, we assume that for all $K' > K$, the claim is true. Then, we have for all $i \notin \mathbf{v}[1]$, $|(0, v_{-i})[1]| = |\mathbf{v}[1]| = K+1$, and induction hypothesis implies that

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) = -\frac{\theta}{n}, \tag{4}$$

Adding Eqs. (3) and (4), and using BB and ETE, we get

$$0 = (K+1)p_1(0, v_{-1}) + \theta - (n-K-1)\frac{\theta}{n}.$$

Simplifying, we get,

$$p_1(0, v_{-1}) = -\frac{\theta}{n}.$$

This shows that if $|(0, v_{-1})[1]| > 1$, then the claim is true.

STEP 2. Let $\mathbf{v}$ be a type profile such that for all $k > 2$ and for all $i \in \mathbf{v}[k]$, we have $v_i = 0$, and $|\mathbf{v}[1]| = 1$ and $|\mathbf{v}[2]| > 1$. In this step, we show that if $\theta = v_i > 0$ for every $i \in \mathbf{v}[2]$, then

$$p_i(0, v_{-i}) = -\frac{\theta}{n}.$$

Suppose $\mathbf{v}[1] = \{1\}$. By step 1,

$$p_1(0, v_{-1}) = -\frac{\theta}{n}. \tag{5}$$

Further, by Property P2, $f_1(\mathbf{v}) = 1$. Further, for all $x_1 \in (\theta, v_1)$, we have $f_1(x_1, v_{-1}) = 1$ and for all $x_1 < \theta$, we have $f_1(x_1, v_{-1}) = 0$—the latter observation follows from the fact that $|(x_1, v_{-1})[1]| > 1$ and Properties P0 and P1. Hence, using Eqs. (2) and (5), we get

$$p_1(\mathbf{v}) = -\frac{\theta}{n} + v_1 - (v_1 - \theta) = (1 - 1/n)\theta. \tag{6}$$

Suppose $|\mathbf{v}[2]| = K$. By Property P2, $f_i(\mathbf{v}) = 0$ for all $i \in \mathbf{v}[2]$. Hence, for each $i \in \mathbf{v}[2]$, Eq. (2) implies that

$$p_i(\mathbf{v}) = p_i(0, v_{-i}). \tag{7}$$

If $K = n - 1$, by adding Eqs. (6) and (7), and using BB and ETE, we get for every $i \in \mathbf{v}[2]$,

$$0 = (n-1)p_i(0, v_{-i}) + (1 - 1/n)\theta.$$

This simplifies to $p_i(0, v_{-i}) = -\frac{\theta}{n}$.

Now, we use induction on K. Suppose the claim is true for all $K' > K$ and $K < n - 1$. By construction, for all $j > 2$ and for all $i \in \mathbf{v}[j]$, $v_i = 0$. We can construct another type profile $\mathbf{v}'$ such that $v'_i = \theta$ and $v'_j = v_j$ for all $j \neq i$. Note that $|\mathbf{v}'[2]| = K + 1$. Hence, induction hypothesis implies that

$$p_i(0, v'_{-i}) = p_i(0, v_{-i}) = p_i(\mathbf{v}) = -\frac{\theta}{n}. \tag{8}$$

Adding Eqs. (6)–(8), and using BB and ETE we get for every $i \in \mathbf{v}[2]$,

$$0 = Kp_i(0, v_{-i}) + (1 - 1/n)\theta - \frac{n - K - 1}{n}\theta.$$

This simplifies to $p_i(0, v_{-i}) = -\frac{\theta}{n}$, as desired.

Step 3 Now, we complete the proof. Pick a $\mathbf{v}$ with $\mathbf{v}[1] = \{1\}$ and $|\mathbf{v}[2]| > 1$. Suppose $v_k = \theta > 0$ for all $k \in \mathbf{v}[2]$. Note that by Step 1, the claim is proved if we show that for all $i \notin \mathbf{v}[1]$, we have $p_i(0, v_{-i}) = -\frac{\theta}{n}$—in this case $(0, v_{-i})$ is a type profile such that $|(0, v_{-i})[1]| = 1$.

Suppose $K = |\mathbf{v}[2]|$. We use induction on K. If $K = n - 1$, the claim follow from step 2. Suppose the claim is true for all $K' > K$. Pick $i \in \mathbf{v}[k]$, where $k > 2$. We can construct a type profile $\mathbf{v}'$ with $v'_i = \theta$ and $v_j = v'_j$ for all $j \neq i$. Since $|\mathbf{v}'[2]| = K + 1$, induction hypothesis implies that

$$p_i(0, v'_{-i}) = p_i(0, v_{-i}) = -\frac{\theta}{n}. \tag{9}$$

Now, at type profile $\mathbf{v}$, we know that $\mathbf{v}[1] = \{1\}$ and $|\mathbf{v}[2]| > 1$. By Property P2, $f_1(\mathbf{v}) = 1$ and for all $x_1 \in (\theta, v_1)$, we have $f_1(x_1, v_{-1}) = 1$. Further, by Properties P0 and P1, $f_1(x_1, v_{-1}) = 0$ for all $x_1 < \theta$. Using these observations and Eq. (2), we get

$$p_1(\mathbf{v}) = p_1(0, v_{-1}) + v_1 - (v_1 - \theta) = -\frac{\theta}{n} + \theta = (1 - 1/n)\theta, \tag{10}$$

where the second equality follows from step 1. Since $f_i(\mathbf{v}) = 0$ for all $i \neq 1$, we can argue the following. For every $i \in \mathbf{v}[2]$, we have

$$p_i(\mathbf{v}) = p_i(0, v_{-i}). \tag{11}$$

For every $i \in \mathbf{v}[k]$, where $k > 2$, using Eq. (9),

$$p_i(\mathbf{v}) = p_i(0, v_{-i}) = -\frac{\theta}{n}. \tag{12}$$

Adding Eqs. (10)–(12), and using ETE we get for every $i \in \mathbf{v}[2]$,

$$0 = Kp_i(0, v_{-i}) + (1 - 1/n)\theta - (n - K - 1)\frac{\theta}{n}.$$

Simplifying, we get $p_i(0, v_{-i}) = -\frac{\theta}{n}$, as desired. □

Now, we complete the proof of Proposition 3. Suppose $(f, \mathbf{p})$ is a satisfactory mechanism and f satisfies Properties P0, P1, and P2. Using Lemma 1, we immediately get that $p_i(0, v_{-i}) = -\frac{v_3}{n}$ if $i \in \{1, 2\}$ and $p_i(0, v_{-i}) = -\frac{v_2}{n}$ if $i \notin \{1, 2\}$. Using these equations, we get $\sum_{i \in N} p_i(0, v_{-i}) = -\frac{1}{n}[(n - 2)v_2 + 2v_3]$. □

Now, we complete the remaining part of Proof of Theorem 1. Assume for contradiction that mechanism $\tilde{\mathcal{M}} \equiv (\tilde{f}, \tilde{\mathbf{p}})$ is a top-only satisfactory mechanism such that for all $\mathbf{v}$, we have

$$W(\mathbf{v}; \tilde{\mathcal{M}}) \geq W(\mathbf{v}, M^*), \tag{13}$$

with strict inequality holding for some $\mathbf{v}$.

Every top-only allocation rule satisfies Property P0. Since f^* satisfies Properties P1 and P2, Eq. (13) implies that $\tilde{f}$ satisfies Properties P1 and P2—this is because an implication of Eq. (13) is that $\tilde{f}$ is efficient at all valuation profiles where f^* is efficient, and f^* is efficient at the profiles mentioned in Properties P1 and P2.

Then, by Proposition 3, we have for all $\mathbf{v}$ with $v_1 \geq v_2 \geq \cdots \geq v_n$,

$$\sum_{i \in N} \tilde{p}_i(0, v_{-i}) = \sum_{i \in N} p_i^*(0, v_{-i}) = -\frac{1}{n}[(n - 2)v_2 + 2v_3]. \tag{14}$$

Note that if $v_2 = v_3$, then Properties P1 and P2 imply that $\tilde{f}_1(\mathbf{v}) = f_1^*(\mathbf{v}) = 1$. Now suppose $v_2 > v_3$. If $v_1 = v_2$, then by revenue equivalence formula and using the fact that $\tilde{f}_1(x_1, v_{-1}) = \tilde{f}_2(x_2, v_{-2}) = 0$ for all $x_1, x_2 < v_1 (= v_2)$, we get

$$\begin{aligned} p_1(\mathbf{v}) &= p_1(0, v_{-1}) + v_1 \tilde{f}_1(\mathbf{v}) \\ p_2(\mathbf{v}) &= p_2(0, v_{-2}) + v_2 \tilde{f}_2(\mathbf{v}) \\ p_j(\mathbf{v}) &= p_j(0, v_{-j}) \quad \forall\, j \notin \{1, 2\}. \end{aligned}$$

Adding and using budget-balance and ETE, we have

$$\sum_{i \in N} p_i(0, v_{-i}) = -2v_1 \tilde{f}_1(\mathbf{v}) = -2v_2 \tilde{f}_1(\mathbf{v}).$$

Using Eq. (14), we get

$$\tilde{f}_1(\mathbf{v}) = \tilde{f}_2(\mathbf{v}) = \frac{1}{2n}\left[(n - 2) + 2\frac{v_3}{v_2}\right] = f_1^*(\mathbf{v}) = f_2^*(\mathbf{v}).$$

Hence, if $v_1 = v_2$ or $v_2 = v_3$, by top-only property $\tilde{f} = f^*$. Since Eq. (13) holds strictly for some $\mathbf{v}$, such a valuation profile must satisfy $v_1 > v_2 > v_3$. By top-only property and Eq. (13), we must have

$$\tilde{f}_1(\mathbf{v}) > f_1^*(\mathbf{v}). \tag{15}$$

But then,

$$\begin{aligned}
0 &= \sum_{i \in N} \tilde{p}_i(\mathbf{v}) \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + \sum_{i \in N} v_i \tilde{f}_i(\mathbf{v}) - \sum_{i \in N} \left[\int_0^{v_i} \tilde{f}_i(x_i, v_{-i} dx_i \right] \\
&\quad \text{(by revenue equivalence)} \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_1 \tilde{f}_1(\mathbf{v}) - \int_{v_2}^{v_1} \tilde{f}_1(x_1, v_{-1}) dx_1 \quad \text{(by top-only property of } \tilde{f}) \\
&\geq \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_1 \tilde{f}_1(\mathbf{v}) - (v_1 - v_2) \tilde{f}_1(\mathbf{v}) \quad \text{(from monotonicity of } \tilde{f}_1) \\
&= \sum_{i \in N} \tilde{p}_i(0, v_{-i}) + v_2 \tilde{f}_1(\mathbf{v}) \\
&> -\frac{1}{n}[(n-2)v_2 + 2v_3] + v_2 f_1^*(\mathbf{v}) \quad \text{(from Eq. (14) and Inequality (15))} \\
&= 0 \quad \text{(by definition of } f^*),
\end{aligned}$$

which is a contradiction.

This completes the proof of Theorem 1.

Proof of Theorem 2

We do the proof with the help of some lemmas.

Lemma 2 *Suppose* f *is a satisfactorily implementable and satisfies Properties* P0, P1, *and* P2. *Then, for all valuation profiles* $\mathbf{v} \in V^n$ *and for* $i \in \mathbf{v}[k]$ *with* $k > 2$, *we have*

$$f_i(\mathbf{v}) = 0.$$

Proof Consider a valuation profile $\mathbf{v}$. Denote the valuation of agents in $\mathbf{v}[k]$ for any k as θ_k. Pick an agent $i \in \mathbf{v}[3]$ and consider a valuation profile $\mathbf{v}'$ as follows: $v'_j = v_j$ if $j \neq i$ and $v'_i = \theta_2$ (in other words, valuation of agent i is increased to second ranked valuation). Note that $|\mathbf{v}'[2]| > 1$ and $i \in \mathbf{v}'[2]$. Hence, by Properties P0, P1, and P2, we have $f_i(\mathbf{v}') = 0$. By monotonicity of f and ETE, we get that $f_i(\mathbf{v}) = 0$ and $f_j(\mathbf{v}) = 0$ for all $j \in \mathbf{v}[3]$.

We now use induction. We assume that at any valuation profile $\mathbf{v}$ and for all $k < K$ and $k \geq 3$, we have $f_j(\mathbf{v}) = 0$ for all $j \in \mathbf{v}[k]$. We now show that $f_j(\mathbf{v}) = 0$ for all $j \in \mathbf{v}[K]$. To do so, we pick an agent $i \in \mathbf{v}[K]$ and construct a valuation profile $\mathbf{v}'$ as follows: $v'_j = v_j$ if $j \neq i$ and $v'_i = \theta_{K-1}$. Since $i \in \mathbf{v}'[K-1]$, by the induction hypothesis, $f_i(\mathbf{v}') = 0$. By monotonicity of f and ETE, we get that $f_i(\mathbf{v}) = 0$ and $f_j(\mathbf{v}) = 0$ for all $j \in \mathbf{v}[K]$. This completes the proof. □

The next lemma uses the following strengthening of Properties P0 and P1.

Definition 10 An allocation rule f satisfies Property C1 if for every $\mathbf{v}$ with $|\mathbf{v}[1]| > 1$, we have $\sum_{i \in \mathbf{v}[1]} f_i(\mathbf{v}) = 1$.

Note that Property C1 implies Properties P0 and P1.

Lemma 3 *Suppose f is a satisfactorily implementable allocation rule satisfying Properties* C1 *and* P2. *Then, for every $v_{-2} \equiv (v_1, v_3, v_4, \ldots, v_n)$ with $v_1 > v_3 \geq v_4 \geq \cdots \geq v_n$, we have*

$$\int_{v_3}^{v_1} f_2(x_2, v_{-2}) dx_2 = \frac{1}{n}.$$

Proof Consider $v_{-2} \equiv (v_1, v_3, v_4, \ldots, v_n)$ with $v_1 > v_3 \geq v_4 \geq \cdots \geq v_n$. Further, consider a type profile $\mathbf{v}'$ such that $v'_2 = v_1$ and $v'_j = v_j$ for all $j \neq 1$. Since C1 implies Properties P0 and P1, Proposition 3 gives

$$\sum_{i \in N} p_i(0, v'_{-i}) = \frac{1}{n}[(n-2)v_1 + 2v_3]. \tag{16}$$

Observing that $|\mathbf{v}'[1]| > 1$ and using Property C1, we get $f_1(\mathbf{v}') + f_2(\mathbf{v}') = 1$. By the revenue equivalence formula, $p_i(\mathbf{v}') = p_i(0, v'_{-i})$ for all $i \notin \{1, 2\}$ and for all $i \in \{1, 2\}$,

$$p_i(\mathbf{v}') = p_i(0, v'_{-i}) + \frac{1}{2} v_1 - \int_{v_3}^{v_1} f_2(x_2, v_{-2}) dx_2,$$

where we used the fact that $v'_j = v_j$ for all $j \neq 2$ and ETE. Using budget-balance, we get that

$$\sum_{i \in N} p_i(0, v'_{-i}) = v_1 - 2 \int_{v_3}^{v_1} f_2(x_2, v_{-2}) dx_2. \tag{17}$$

Using Eqs. (16) and (17), and simplifying we get

$$\int_{v_3}^{v_1} f_2(x_2, v_{-2}) dx_2 = \frac{1}{n}(v_1 - v_3).$$

□

We are now ready to complete the proof of Theorem 2.

Proof of Theorem 2. The fact that the MGL mechanism is satisfactory is routine to check—BB and ETE is clear, and for DSIC, one can either do a direct check of incentive constraints or verify that the revenue equivalence formula holds.

To prove that the MGL mechanism maximizes utilitarian welfare across all satisfactory mechanisms, suppose there is a satisfactory mechanism $M \equiv (f, \mathbf{p})$ such that

$$W(\mathbf{v}; M) \geq W(\mathbf{v}; M^{G'}) \quad \forall\, \mathbf{v}, \tag{18}$$

with strict inequality satisfying for some $\mathbf{v}$. This implies that f is efficient at all valuation profiles where $f^{G'}$ is efficient. Then f must satisfy Properties C1 and P2.

Choose a type profile $\mathbf{v}$ with $v_1 \geq v_2 \geq \cdots \geq v_n$. Note that if $v_1 = v_2$ or $v_2 = v_3$, then Properties C1 and P2 imply that $f^{G'}(\mathbf{v}) = f(\mathbf{v})$. So, we consider $\mathbf{v}$ such that $v_1 > v_2 > v_3 \geq v_4 \geq \cdots \geq v_n$.

Now, for any $x_2 \in (v_3, v_1)$, Lemma 2 implies that $f_j(x_2, v_{-2}) = 0$ for all $j > 2$. Hence, Eq. (18) implies that

$$v_1 f_1(x_2, v_{-2}) + x_2 f_2(x_2, v_{-2}) \geq v_1(1 - 1/n) + x_2/n.$$

Using $f_1(x_2, v_{-2}) + f_2(x_2, v_{-2}) \leq 1$, we simplify this to get

$$(v_1 - x_2) f_1(x_2, v_{-2}) \geq (v_1 - x_2)(1 - 1/n).$$

But $v_1 > x_2$ implies that $f_1(x_2, v_{-2}) \geq 1 - 1/n$ and $f_2(x_2, v_{-2}) \leq 1/n$. Using Lemma 3 along with monotonicity of f_2, we get $f_2(x_2, v_{-2}) = 1/n$, and hence, $f_1(x_2, v_{-2}) = 1 - 1/n$ for all $x_2 \in (v_3, v_1)$. This implies that $f_1(\mathbf{v}) = 1 - 1/n$ and $f_2(\mathbf{v}) = 1/n$ as desired.

References

Arrow K (1979) The property rights doctrine and demand revelation under incomplete information. In: Boskin M (ed) Economics and human welfare. Academic Press, New York, pp 23–39

Cavallo R (2006) Optimal decision-making with minimal waste: strategyproof redistribution of VCG payments. In: Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems. ACM, New York, pp 882–889

Cramton P, Gibbons R, Klemperer P (1987) Dissolving a partnership efficiently. Econometrica 55:615–632

d'Aspremont C, Gérard-Varet L-A (1979) Incentives and incomplete information. J Public Econ 11:25–45

Drexl M, Kleiner A (2015) Optimal private good allocation: the case for a balanced budget. Games Econ Behav 94:169–181

Green JR, Laffont J-J (1979) Incentives in public decision making. North-Holland, Amsterdam

Guo M, Conitzer V (2009) Worst-case optimal redistribution of VCG payments in multi-unit auctions. Games Econ Behav 67:69–98

Guo M, Naroditskiy V, Conitzer V, Greenwald A, Jennings NR (2011) Budget-balanced and nearly efficient randomized mechanisms: public goods and beyond. In: Chen N, Elkind E, Koutsoupias E (eds) Internet and network economics. WINE 2011. Lecture Notes in Computer Science, vol 7090. Springer, Berlin, Heidelberg, pp 158–169

Hashimoto K (2015) Strategy-proof rule in probabilistic allocation problem of an indivisible good and money. Working Paper, Osaka University

Holmström B (1979) Groves' scheme on restricted domains. Econometrica 47:1137–1144

Hurwicz L, Walker M (1990) On the generic nonoptimality of dominant-strategy allocation mechanisms: a general theorem that includes pure exchange economies. Econometrica 58:683–704

Krishna V (2009) Auction theory. Academic press, New York

Laffont J-J, Maskin E (1980) A differential approach to dominant strategy mechanisms. Econometrica 48:1507–1520

Long Y, Mishra D, Sharma T (2017) Balanced ranking mechanisms. Games Econ Behav 105:9–39

Malmquist S (1950) On a property of order statistics from a rectangular distribution. Scand Actuar J 1950:214–222

Moulin H (2009) Almost budget-balanced VCG mechanisms to assign multiple objects. J Econ Theory 144:96–119

Moulin H (2010) Auctioning or assigning an object: some remarkable VCG mechanisms. Soc Choice Welf 34:193–216

Myerson RB (1981) Optimal auction design. Math Oper Res 6:58–73

Nath S, Sandholm T (2016) Efficiency and budget balance. In: Cai Y, Vetta A (eds) Web and internet economics. WINE 2016. Lecture Notes in Computer Science, vol 10123. Springer, Berlin, Heidelberg, pp 369–383

Reiss R-D (2012) Approximate distributions of order statistics: with applications to nonparametric statistics. Springer, Berlin

Sprumont Y (2013) Constrained-optimal strategy-proof assignment: beyond the Groves mechanisms. J Econ Theory 148:1102–1121

Walker M (1980) On the nonexistence of a dominant strategy mechanism for making optimal public decisions. Econometrica 48:1521–1540